Unit 1: Classification of signals and systems

1.1 Signal

Signal is one that carries information and is defined as a physical quantity that varies with one or more independent variable.

Example: Music, speech

1.2 Classification of signals

1.2.1 Analog and Digital signal

Analog signal:

A signal that is defined for every instants of time is known as analog signal. Analog signals are continuous in amplitude and continuous in time. It is denoted by x(t). It is also called as **Continuous time** signal. Example for Continuous time signal is shown in Fig 1.1

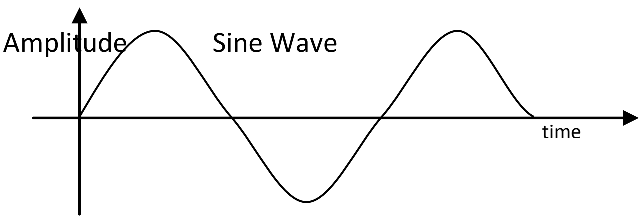


Fig 1.1 Continuous time signal

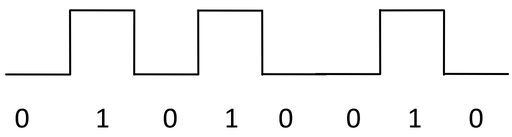


Fig 1.2 Digital Signal

Digital signal:

The signals that are discrete in time and quantized in amplitude is called digital signal (Fig 1.2)

1.2.2 Continuous time and discrete time signal

Continuous time signal:

A signal that is defined for every instants of time is known as continuous time signal. Continuous time signals are continuous in amplitude and continuous in time. It is denoted by x(t) and shown in Fig 1.1

Discrete time signal:

A signal that is defined for discrete instants of time is known as discrete time signal. Discrete time signals are continuous in amplitude and discrete in time. It is also obtained by sampling a continuous time signal. It is denoted by x(n) and shown in Fig 1.3

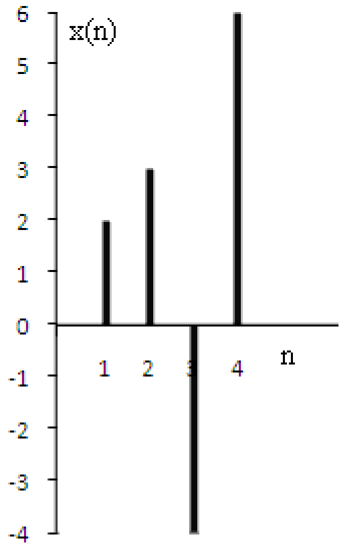


Fig 1.3 Discrete time signal

1.2.3 Even (symmetric) and Odd (Anti-symmetric) signal

Continuous domain:

Even signal:

A signal that exhibits symmetry with respect to t=0 is called even signal
Even signal satisfies the condition $x(t) = x(-t)$

Odd signal:

A signal that exhibits anti-symmetry with respect to t=0 is called odd signal
Odd signal satisfies the condition $x(t) = -x(-t)$

Even part $\boldsymbol{x_e(t)}$ and Odd part $\boldsymbol{x_0(t)}$ of continuous time signal $\boldsymbol{x(t)}$:

Even part $x_e(t) = \frac{1}{2}[x(t) + x(-t)]$

Odd part $x_o(t) = \frac{1}{2}[x(t) - x(-t)]$

Discrete domain:

Even signal:

A signal that exhibits symmetry with respect to n=0 is called even signal
Even signal satisfies the condition $x(n) = x(-n)$.

Odd signal:

A signal that exhibits anti-symmetry with respect to n=0 is called odd signal Odd signal satisfies the condition $x(n) = -x(-n)$.

Even part $\boldsymbol{x_e(n)}$ and Odd part $\boldsymbol{x_0(n)}$ of discrete time signal $\boldsymbol{x(n)}$:

Even part $x_e(n) = \frac{1}{2}[x(n) + x(-n)]$

Odd part $x_o(n) = \frac{1}{2}[x(n) - x(-n)]$

1.2.4 Periodic and Aperiodic signal

Periodic signal:

A signal is said to periodic if it repeats again and again over a certain period of time.

Aperiodic signal:

A signal that does not repeat at a definite interval of time is called aperiodic signal.

Continuous domain:

A Continuous time signal is said to **periodic** if it satisfies the condition

$$x(t) = x(t+T) \;\; where\ T\ is\ fundamental\ time\ period$$

If the above condition is not satisfied then the signal is said to be **aperiodic**

Fundamental time period $\mathbf{T} = \frac{\mathbf{2\pi}}{\Omega}$, where Ω is fundamental angular frequency in rad/sec

Discrete domain:

A Discrete time signal is said to **periodic** if it satisfies the condition

$$x(n) = x(n+N) \;\; where\ N\ is\ fundamental\ time\ period$$

If the above condition is not satisfied then the signal is said to be **aperiodic**

Fundamental time period $\mathbf{N} = \frac{\mathbf{2\pi m}}{\boldsymbol{\omega}}$, where ω is fundamental angular frequency in rad/sec, m is smallest positive integer that makes N as positive integer

1.2.5 Energy and Power signal

Energy signal:

The signal which has finite energy and zero average power is called energy signal. The non periodic signals like exponential signals will have constant energy and so non periodic signals are energy signals.

i.e., For energy signal, $0 < E < \infty \quad and \quad P = 0$

For Continuous time signals,

$$Energy\ E = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt$$

For Discrete time signals,

$$Energy\ E = \lim_{N\to\infty} \sum_{n=-N}^{N} |x(n)|^2$$

Power signal:

The signal which has finite average power and infinite energy is called power signal. The periodic signals like sinusoidal complex exponential signals will have constant power and so periodic signals are power signals.

i.e., For power signal, $0 < P < \infty \quad and \quad E = \infty$

For Continuous time signals,

$$Average\ power\ P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

For Discrete time signals,

$$Average\ power\ P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

1.2.6 Deterministic and Random signals

Deterministic signal:

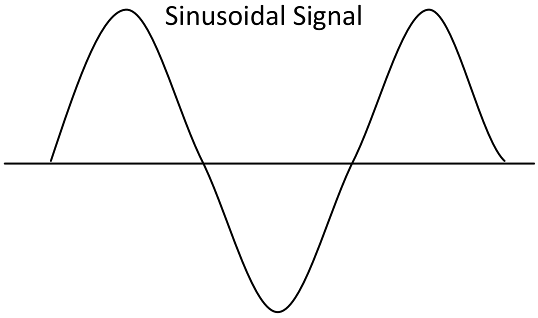
A signal is said to be deterministic if there is no uncertainity over the signal at any instant of time i.e., its instantaneous value can be predicted. It can be represented by mathematical equation.

Example: sinusoidal signal

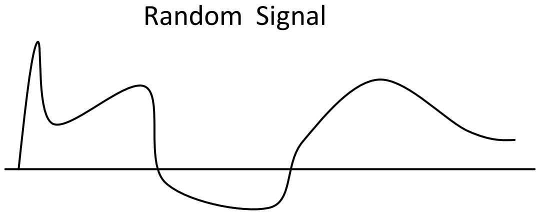
Random signal (Non-Deterministic signal):

A signal is said to be random if there is uncertainity over the signal at any instant of time i.e., its instantaneous value cannot be predicted. It cannot be represented by mathematical equation.

Example: noise signal



Deterministic signal



Random signal

1.2.7 Causal and Non-causal signal

Continuous domain:

Causal signal:

A signal is said to be causal if it is defined for t≥0.

$$i.e., \quad x(t) = 0 \quad for\ t < 0$$

Non-causal signal:

A signal is said to be non-causal, if it is defined for t< 0 or for both $t < 0$ and $t \geq 0$

$$i.e., \quad x(t) \neq 0 \quad for\ t < 0$$

When a non-causal signal is defined only for t<0, it is called as **anti-causal signal**

Discrete domain:

Causal signal:

A signal is said to be causal, if it is defined for n≥0.

$$i.e., \quad x(n) = 0 \quad for\ n < 0$$

Non-causal signal:

A signal is said to be non-causal, if it is defined for n< 0 or for both n < 0 *and* $n \geq 0$

$$i.e., \quad x(n) \neq 0 \quad for\ n < 0$$

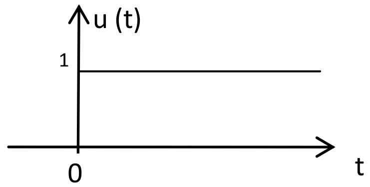
When a non-causal signal is defined only for n<0, it is called as **anti-causal signal**

1.3 Basic(Elementary or Standard) continuous time signals

1.3.1 Step signal

Unit Step signal is defined as

$$u(t) = 1 \ \ for\ t \geq 0$$
$$= 0\ for\ t < 0$$

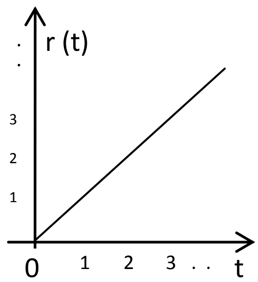


Unit step signal

1.3.2 Ramp signal

Unit ramp signal is defined as

$$r(t) = t \ \ for\ t \geq 0$$
$$= 0\ for\ t < 0$$

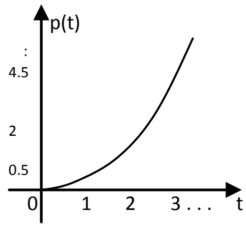


Unit ramp signal

1.3.3 Parabolic signal

Unit Parabolic signal is defined as

$$x(t) = \frac{t^2}{2} \;\; for\; t \geq 0$$
$$= 0\; for\; t < 0$$



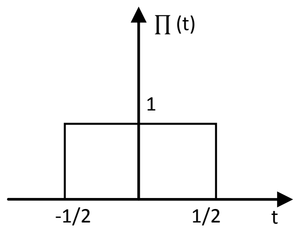
Unit Parabolic signal

Relation between Unit Step signal, Unit ramp signal and Unit Parabolic signal:

- Unit ramp signal is obtained by integrating unit step signal
$$i.e., \int u(t)dt = \int 1dt = t = r(t)$$
- Unti Parabolic signal is obtained by integrating unit ramp signal
$$i.e., \int r(t)dt = \int tdt = \frac{t^2}{2} = p(t)$$
- Unit step signal is obtained by differentiating unit ramp signal
$$i.e., \frac{d}{dt}\big(r(t)\big) = \frac{d}{dt}(t) = 1 = u(t)$$
- Unit ramp signal is obtained by differentiating unit Parabolic signal
$$i.e., \frac{d}{dt}\big(p(t)\big) = \frac{d}{dt}\left(\frac{t^2}{2}\right) = \frac{1}{2}(2t) = t = r(t)$$

1.3.4 Unit Pulse signal is defined as

$$\prod(t) = 1 \;\; for\; |t| \leq \frac{1}{2}$$
$$= 0\; elsewhere$$

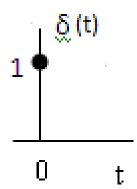


Unit Pulse signal

1.3.5 Impulse signal

Unit Impulse signal is defined as

$$\delta(t) = 0\; for\; t \neq 0$$
$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$



Unit Impulse signal

Properties of Impulse signal:

Property 1:

$$\int_{-\infty}^{\infty} \boldsymbol{x(t)\delta(t)\,dt = x(0)}$$

Proof:

$$\int_{-\infty}^{\infty} x(t)\delta(t)\,dt = x(0)\delta(0) = x(0) \qquad [\because \delta(t)\ exists\ only\ at\ t = 0\ and\ \delta(0) = 1]$$

Thus proved

Property 2:

$$\int_{-\infty}^{\infty} \boldsymbol{x(t)\delta(t - t_0)\,dt = x(t_0)}$$

Proof:

$$\int_{-\infty}^{\infty} x(t)\delta(t - t_0)\,dt = x(t_0)\delta(t_0 - t_0) = x(t_0)\delta(0) = x(t_0)$$

$$[\because \delta(t - t_0)\ exists\ only\ at\ t = t_0\ and\ \delta(0) = 1]$$

Thus proved

1.3.6 Sinusoidal signal

Cosinusoidal signal is defined as

$$x(t) = Acos(\Omega t + \Phi)$$

Sinusoidal signal is defined as

$$x(t) = Asin(\Omega t + \Phi)$$

where $\Omega = 2\pi f = \frac{2\pi}{T}$ and Ω is angular frequency in rad/sec

f is frequency in cycles/sec or Hertz and

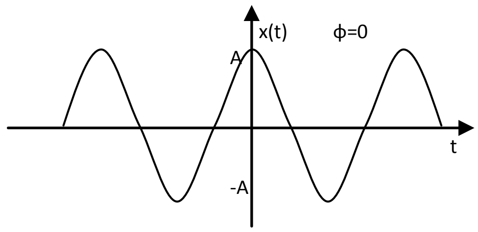
A is amplitude

T is time period in seconds

Φ is phase angle in radians

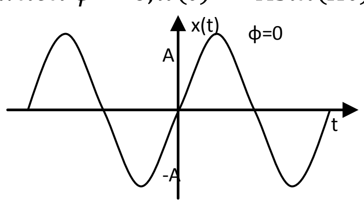
Cosinusoidal signal

$$when\ \phi = 0, x(t) = Acos(\Omega t)$$



Sinusoidal signal

$$when\ \phi = 0, x(t) = Asin(\Omega t)$$



Sinusoidal signal

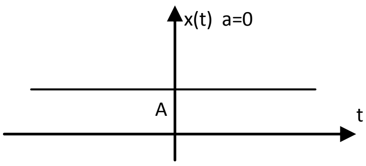
Cosinusoidal signal

1.3.7 Exponential signal

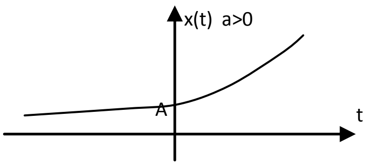
Real Exponential signal is defined as $x(t) = Ae^{at}$

where A is amplitude

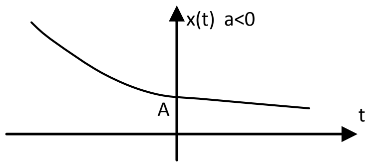
Depending on the value of 'a' we get dc signal or growing exponential signal or decaying exponential signal



DC signal



Exponentially growing signal



Exponentially decaying signal

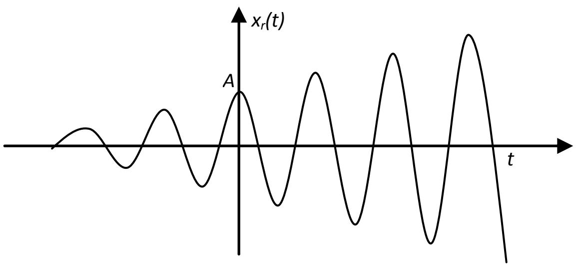
Complex exponential signal is defined as $x(t) = Ae^{st}$

where A is amplitude, s is complex variable and $s = \sigma + j\Omega$

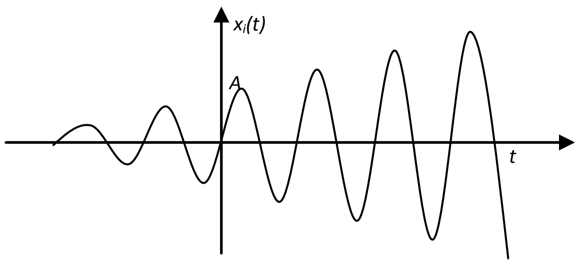
$$x(t) = Ae^{st} = Ae^{(\sigma+j\Omega)t} = Ae^{\sigma t}e^{j\Omega t} = Ae^{\sigma t}(cos\Omega t + jsin\Omega t)$$

$when\ \sigma = +ve, then\ x(t) = Ae^{\sigma t}(cos\Omega t + jsin\Omega t),$

$where\ x_r(t) = Ae^{\sigma t}cos\Omega t\ and\ x_i(t) = Ae^{\sigma t}sin\Omega t$



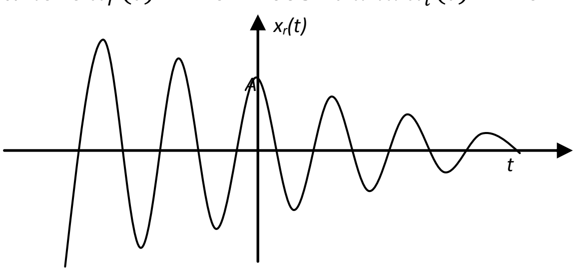
Exponentially growing Cosinusoidal signal

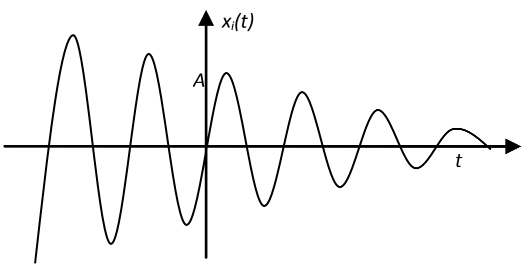


Exponentially growing sinusoidal signal

$when\ \sigma = -ve, then\ x(t) = Ae^{-\sigma t}(cos\Omega t + jsin\Omega t),$

$where\ x_r(t) = Ae^{-\sigma t}cos\Omega t\ and\ x_i(t) = Ae^{-\sigma t}sin\Omega t$





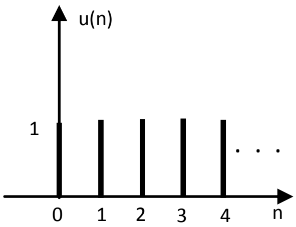
Exponentially decaying Cosinusoidal signal	Exponentially decaying sinusoidal signal

1.4 Basic(Elementary or Standard) Discrete time signals

1.4.1 Step signal

Unit Step signal is defined as

$$u(n) = 1 \;\; for\, n \geq 0$$
$$\quad = 0 \; for\, n < 0$$

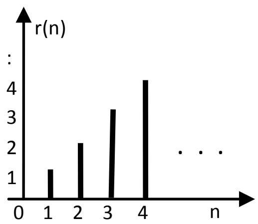


Unit step signal

1.4.2 Unit Ramp signal

Unit Ramp signal is defined as

$$r(n) = n \;\; for\, n \geq 0$$
$$\quad = 0 \quad for\, n < 0$$

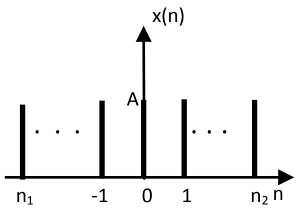


Unit Ramp signal

1.4.3 Pulse signal (Rectangular pulse function)

Pulse signal is defined as

$$x(n) = A \;\; for\, n_1 \leq n \leq n_2$$
$$\quad = 0 \; elsewhere$$

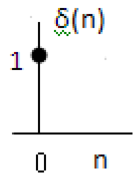


Pulse signal

1.4.4 Unit Impulse signal

Unit Impulse signal is defined as

$$\delta(n) = 1 \; for \; n = 0$$
$$\delta(n) = 0 \; for \; n \neq 0$$



Unit Impulse signal

1.4.5 Sinusoidal signal

Cosinusoidal signal is defined as

$$x(n) = Acos(\omega n)$$

Sinusoidal signal is defined as

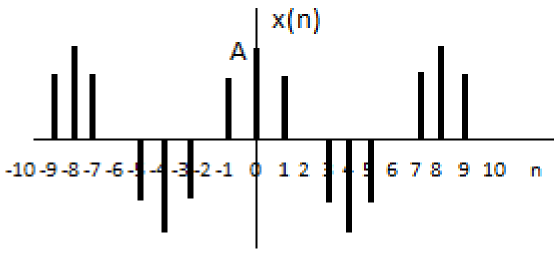
$$x(n) = Asin(\omega n)$$

where $\omega = 2\pi f = \frac{2\pi}{N} m$ and ω is frequency in radians/sample

m is smallest integer

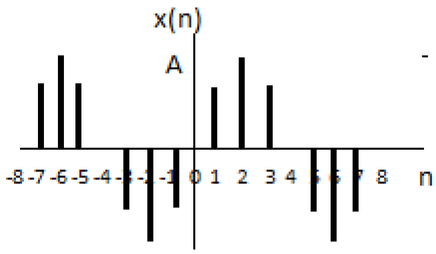
f is frequency in cycles/sample, A is amplitude

Cosinusoidal signal



Cosinusoidal signal

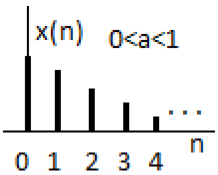
Sinusoidal signal



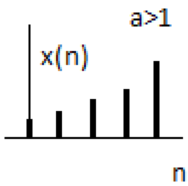
Sinusoidal signal

1.4.6 Exponential signal

Real Exponential signal is defined as $x(n) = a^n \; for \; n \geq 0$



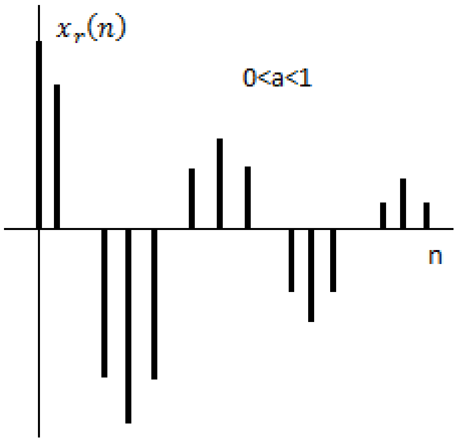
Decreasing exponential signal



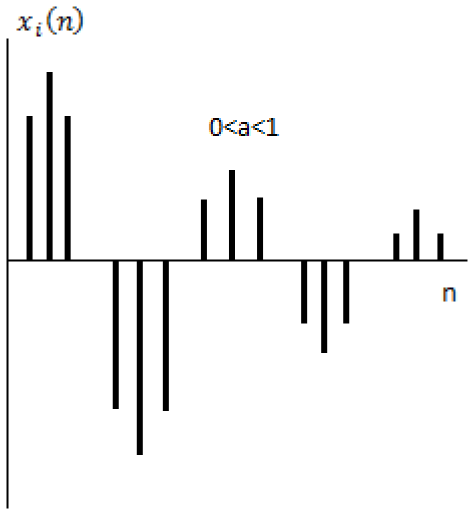
Increasing exponential signal

Complex Exponential signal is defined as $x(n) = a^n e^{j(\omega_0 n)} = a^n[cos\omega_0 n + jsin\omega_0 n]$

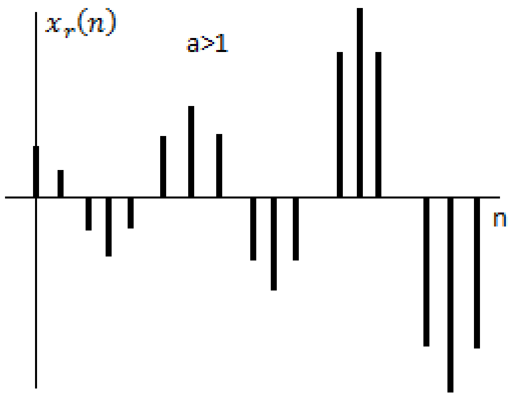
$$where \; x_r(n) = a^n cos\omega_0 n \; and \; x_i(n) = a^n sin\omega_0 n$$



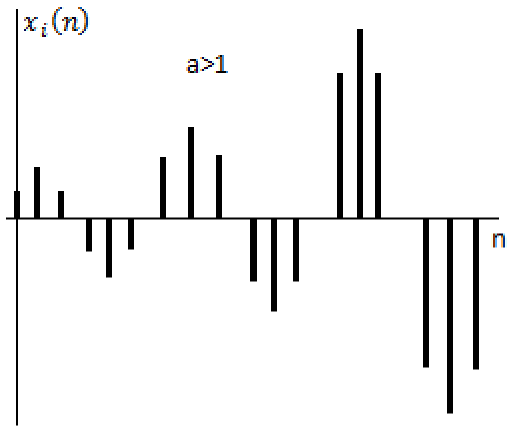
Exponentially decreasing Cosinusoidal signal



Exponentially decreasing sinusoidal signal



Exponentially growing Cosinusoidal signal



Exponentially growing sinusoidal signal

1.5 Classification of System

- Continuous time and Discrete time system
- Linear and Non-Linear system
- Static and Dynamic system
- Time invariant and Time variant system
- Causal and Non-Causal system
- Stable and Unstable system

1.5.1 Continuous time and Discrete time system

Continuous time system:

Continuous time system operates on a continuous time signal (input or excitation) and produces another continuous time signal (output or response) which is shown in Fig 1.84. The signal $x(t)$ is transformed by the system into signal $y(t)$, this transformation can be expressed as,

$$\text{Response } y(t) = T\{x(t)\}$$

where $x(t)$ is input signal, $y(t)$ is output signal, and T denotes transformation

$$x(t) \longrightarrow \boxed{T} \longrightarrow y(t)$$

Fig 1.84 Representation of continuous time system

Discrete time system:

Discrete time system operates on a discrete time signal (input or excitation) and produces another discrete time signal (output or response) which is shown in Fig 1.85.

The signal $x(n)$ is transformed by the system into signal $y(n)$, this transformation can be expressed as,

$$\text{Response } y(n) = T\{x(n)\}$$

where x(n) is input signal, y(n) is output signal, and T denotes transformation

$$x(n) \longrightarrow \boxed{T} \longrightarrow y(n)$$

Fig 1.85 Representation of discrete time system

1.5.2 Linear system and Non Linear system

Continuous time domain:

Linear system:

A system is said to be linear if it obeys superposition theorem. Superposition theorem states that the response of a system to a weighted sum of the signals is equal to the corresponding weighted sum of responses to each of the individual input signals.

Condition for Linearity:

$$T[ax_1(t) + bx_2(t)] = ay_1(t) + by_2(t)$$

where $y_1(t)$ and $y_2(t)$ are the responses of $x_1(t)$ and $x_2(t)$ respectively

Non Linear system:

A system is said to be Non linear if it does not obeys superposition theorem.

$$i.e., T[ax_1(t) + bx_2(t)] \neq ay_1(t) + by_2(t)$$

where $y_1(t)$ and $y_2(t)$ are the responses of $x_1(t)$ and $x_2(t)$ respectively

Discrete time domain:

Linear system:

A system is said to be linear if it obeys superposition theorem. Superposition theorem states that the response of a system to a weighted sum of the signals is equal to the corresponding weighted sum of responses to each of the individual input signals.

Condition for Linearity:

$$T[ax_1(n) + bx_2(n)] = ay_1(n) + by_2(n)$$

where $y_1(n)$ and $y_2(n)$ are the responses of $x_1(n)$ and $x_2(n)$ respectively

Non Linear system:

A system is said to be Non linear if it does not obeys superposition theorem.

$$i.e., T[ax_1(n) + bx_2(n)] \neq ay_1(n) + by_2(n)$$

where $y_1(n)$ and $y_2(n)$ are the responses of $x_1(n)$ and $x_2(n)$ respectively

1.5.3 Static (Memoryless) and Dynamic (Memory) system

Continuous time domain:

Static system:

A system is said to be memoryless or static if the response of the system is due to present input alone.

Example: $y(t) = 2x(t)$

$y(t) = x^2(t) + x(t)$

Dynamic system:

A system is said to be memory or dynamic if the response of the system depends on factors other than present input also.

Example: $y(t) = 2x(t) + x(-t)$

$y(t) = x^2(t) + x(2t)$

Discrete time domain:

Static system:

A system is said to be memoryless or static if the response of the system is due to present input alone.

Example: $y(n) = x(n)$

$y(n) = x^2(n) + 3x(n)$

Dynamic system:

A system is said to be memory or dynamic if the response of the system depends on factors other than present input also.

Example: $y(n) = 2x(n) + x(-n)$

$y(n) = x^2(1 - n) + x(2n)$

1.5.4 Time invariant (Shift invariant) and Time variant (Shift variant) system

Continuous time domain:

Time invariant system:

A system is said to time invariant if the relationship between the input and output does not change with time.

If $y(t) = T[x(t)]$

Then $T[x(t - t_0)] = y(t - t_0)$ should be satisfied for the system to be time invariant

Time variant system:

A system is said to time variant if the relationship between the input and output changes with time.

If $y(t) = T[x(t)]$

Then $T[x(t - t_0)] \neq y(t - t_0)$ should be satisfied for the system to be time variant

Discrete time domain:

Time invariant system:

A system is said to time invariant if the relationship between the input and output does not change with time.

If $y(n) = T[x(n)]$

Then $T[x(n-n_0)] = y(n-n_0)$ should be satisfied for the system to be time invariant

Time variant system:

A system is said to time variant if the relationship between the input and output changes with time.

If $y(n) = T[x(n)]$

Then $T[x(n-n_0)] \neq y(n-n_0)$ should be satisfied for the system to be time variant

1.5.5 Causal and Non-Causal system

Continuous time domain:

Causal system:

A system is said to be causal if the response of a system at any instant of time depends only on the present input, past input and past output but does not depends upon the future input and future output.

Example: $y(t) = 3x(t) + x(t-1)$

A system is said to be causal if impulse response $h(t)$ is zero for negative values of t

i.e., $h(t) = 0 \; for \; t < 0$

Non-Causal system:

A system is said to be Non-causal if the response of a system at any instant of time depends on the future input and also on the present input, past input, past output.

Example: $y(t) = x(t+2) + x(t-1)$

$y(t) = x(-t) + x(t+4)$

A system is said to be non-causal if impulse response $h(t)$ is non-zero for negative values of t

i.e., $h(t) \neq 0 \; for \; t < 0$

Discrete time domain:

Causal system:

A system is said to be causal if the response of a system at any instant of time depends only on the present input, past input and past output but does not depends upon the future input.

Example: $y(n) = 3x(n) + x(n-1)$

A system is said to be causal if impulse response h(n) is zero for negative values of n

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A system is said to be Non-causal if the response of a system at any instant of time depends on the future input and also on the present input, past input, past output.

Example: $y(n) = x(n+2) + x(n-1)$

$$y(n) = x(-n) + x(n+4)$$

A system is said to be non-causal if impulse response $h(n)$ is non-zero for negative values of n

i.e., $h(n) \neq 0$ for $n < 0$

1.5.6 Stable and Unstable system

Continuous time domain:

A system is said to be **stable** if and only if it satisfies the BIBO stability criterion.

BIBO stable condition:

- Every bounded input yields bounded output.

$i.e.,$ if $0 < x(\mathrm{t}) < \infty$ $then$ $0 < y(\mathrm{t}) < \infty$ should be satisfied for the system to be stable

- Impulse response should be absolutely integrable

$$i.e., 0 < \int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$$

If the BIBO stable condition is not satisfied, then the system is said to be **unstable** system

Discrete time domain:

A system is said to be **stable** if and only if it satisfies the BIBO stability criterion.

BIBO stable condition:

- Every bounded input yields bounded output.
- Impulse response should be absolutely summable

$$i.e., 0 < \sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

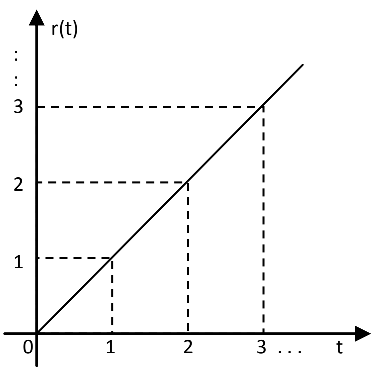
If the BIBO stable condition is not satisfied, then the system is said to be **unstable** system

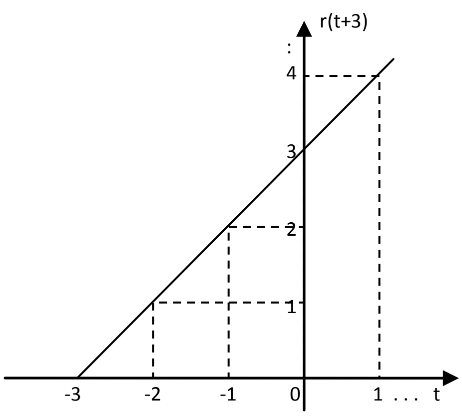
1.6 Solved Problems

1. Draw $r(t+3)$, where r(t) is ramp signal

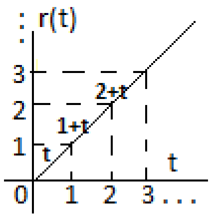
Solution:

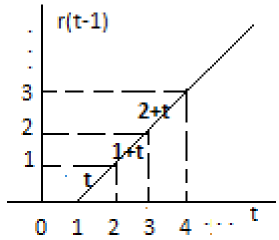
$r(t) = t;\ \ t \geq 0$

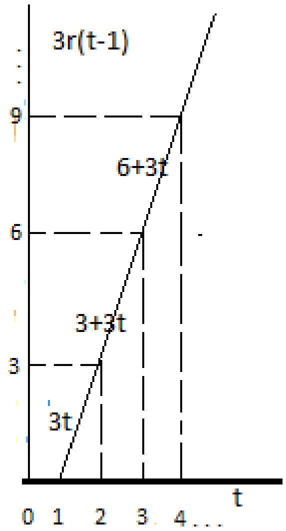


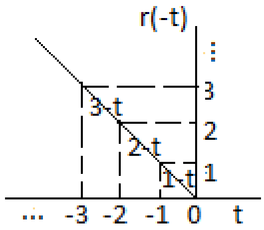


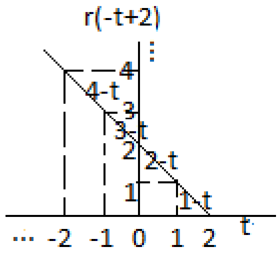
2. Sketch $x(t) = 3r(t-1) + r(-t+2)$











$$
\begin{aligned}
x(t) &= 3r(t-1) + r(-t+2) \\
&= 0 + 4 - t \;\; for -2 \le t \le -1 \\
&= 0 + 3 - t \; for -1 \le t \le 0 \\
&= 0 + 2 - t \; for \; 0 \le t \le 1 \\
&= 3t + 1 - t \; for \; 1 \le t \le 2 \\
&= 3 + 3t + 0 \; for \; 2 \le t \le 3 \\
&= 6 + 3t + 0 \; for \; 3 \le t \le 4
\end{aligned}
$$

and so on

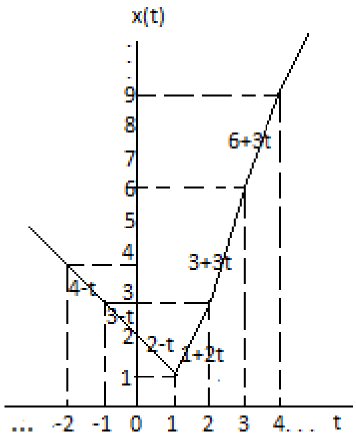
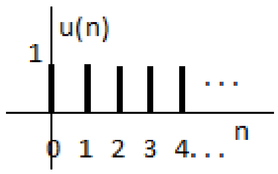


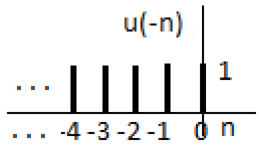
Fig 1.163

3. Draw time reversal signal of unit step signal

Solution:

$u(n) = 1; \;\; n \ge 0$





4. Check whether the following is periodic or not. If periodic, determine fundamental time period

a. $x(t) = 2\cos(5t+1) - \sin(4t)$

Here $\Omega_1 = 5,\ \Omega_2 = 4$

$$T_1 = \frac{2\pi}{\Omega_1} = \frac{2\pi}{5} = \frac{2\pi}{5}$$
$$T_2 = \frac{2\pi}{\Omega_2} = \frac{2\pi}{4} = \frac{\pi}{2}$$
$$\frac{T_1}{T_2} = \frac{\frac{2\pi}{5}}{\frac{\pi}{2}} = \frac{4}{5}\text{(It is rational number)}$$

Hence $x(t)$ is **periodic**

$$T = 5T_1 = 4T_2 = 2\pi$$

$\therefore\ x(t)$ is **periodic** with period $\mathbf{2\pi}$

b. $x(n) = 3\cos 4\pi n + 2\sin \pi n$

Here $\omega_1 = 4\pi,\ \omega_2 = \pi$

$$N_1 = \frac{2\pi m}{\omega_1} = \frac{2\pi m}{4\pi} = \frac{m}{2}$$

$N_1 = 1$ (taking $m = 2$)

$$N_2 = \frac{2\pi m}{\omega_2} = \frac{2\pi m}{\pi} = 2m$$

$N_2 = 2$ (taking $m = 1$)

$$N = LCM(1,2) = 2$$

Hence x(n) $\therefore x(n)$ is **periodic** with period **2**

5. Determine whether the signals are energy or power signal

$$x(t) = e^{-3t}u(t)$$

$$\textbf{Energy } E_\infty = \lim_{T\to\infty}\int_{-T}^{T}|x(t)|^2dt = \lim_{T\to\infty}\int_{0}^{T}|e^{-3t}|^2dt = \lim_{T\to\infty}\int_{0}^{T}e^{-6t}\,dt = \lim_{T\to\infty}\left[\frac{e^{-6t}}{-6}\right]_0^T$$
$$= \lim_{T\to\infty}\left[\frac{e^{-6T}}{-6} - \left[\frac{e^{-0}}{-6}\right]\right] = \frac{\mathbf{1}}{\mathbf{6}} < \infty \qquad \because e^{-\infty} = 0, e^{-0} = 1$$

$$\textbf{\textit{Power }} P_\infty = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|x(t)|^2 dt = \lim_{T\to\infty}\frac{1}{2T}\int_{0}^{T}|e^{-3t}|^2 dt = \lim_{T\to\infty}\frac{1}{2T}\int_{0}^{T} e^{-6t}\, dt$$

$$= \lim_{T\to\infty}\frac{1}{2T}\left[\frac{e^{-6t}}{-6}\right]_0^T = \lim_{T\to\infty}\frac{1}{2T}\left[\frac{e^{-6T}}{-6} - \frac{e^{-0}}{-6}\right] = \lim_{T\to\infty}\frac{1}{2T}\left[\frac{1}{6}\right] = \mathbf{0} \qquad \because e^{-\infty} = 0, e^{-0} = 1, \frac{1}{\infty} = 0$$

Since **energy** value is **finite** and average **power** is **zero,** the given signal is an **energy** signal.

6. Determine whether the signals are energy or power signal

$$\boldsymbol{x(n) = e^{j\left(\frac{\pi n}{4}+\frac{\pi}{2}\right)}}$$

$$\textbf{\textit{Energy }} E_\infty = \lim_{N\to\infty}\sum_{n=-N}^{N}|x(n)|^2 = \lim_{N\to\infty}\sum_{n=-N}^{N}\left|e^{j\left(\frac{\pi n}{4}+\frac{\pi}{2}\right)}\right|^2 = \lim_{N\to\infty}\sum_{n=-N}^{N} 1^2 = \lim_{N\to\infty} 2N+1 = \infty$$

$$\because \left|e^{j(\omega n+\theta)}\right| = 1 \ \ and \ \sum_{n=-N}^{N} 1 = 2N+1$$

$$\textbf{\textit{Average power }} P_\infty = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}|x(n)|^2 = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}\left|e^{j\left(\frac{\pi n}{4}+\frac{\pi}{2}\right)}\right|^2$$

$$= \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}1^2 = \lim_{N\to\infty}\frac{1}{2N+1}(2N+1) = \mathbf{1}$$

Since **energy** value is **infinite** and average **power** is **finite,** the given signal is **power** signal

7. Determine whether the following systems are linear or not

$$\boldsymbol{\frac{dy(t)}{dt} + ty(t) = x^2(t)}$$

Output due to weighted sum of inputs:

$$\frac{d[ay_1(t)+by_2(t)]}{dt} + t[ay_1(t)+by_2(t)] = [ax_1(t)+bx_2(t)]^2 \ldots (1)$$

Weighted sum of outputs:

For input $x_1(t)$:

$$\frac{dy_1(t)}{dt} + ty_1(t) = {x_1}^2(t) \ldots (2)$$

For input $x_2(t)$:

$$\frac{dy_2(t)}{dt} + ty_2(t) = {x_2}^2(t) \ldots (3)$$

$$(2)\times a + (3)\times b \Rightarrow a\frac{dy_1(t)}{dt} + aty_1(t) + b\frac{dy_2(t)}{dt} + bty_2(t) = a{x_1}^2(t) + b{x_2}^2(t) \ldots (4)$$

$$(1) \neq (4)$$

The given system is **Non-Linear**

8. Determine whether the following systems are linear or not

$\boldsymbol{y(n) = x(n-2) + x(n^2)}$

Output due to weighted sum of inputs:

$$y_3(n) = ax_1(n-2) + bx_2(n-2) + ax_1(n^2) + bx_2(n^2)$$

Weighted sum of outputs:

For input $x_1(n)$:

$$y_1(n) = x_1(n-2) + x_1(n^2)$$

For input $x_2(n)$:

$$y_2(n) = x_2(n-2) + x_2(n^2)$$

$$ay_1(n) + by_2(n) = ax_1(n-2) + ax_1(n^2) + bx_2(n-2) + bx_2(n^2)$$

$$\because y_3(n) = ay_1(n) + by_2(n)$$

9. Determine whether the following systems are static or dynamic

$\boldsymbol{y(t) = x(2t) + 2x(t)}$

$$y(0) = x(0) + 2x(0) \Rightarrow \text{ present inputs}$$

$$y(-1) = x(-2) + 2x(-1) \Rightarrow \text{ past and present inputs}$$

$$y(1) = x(2) + 2x(1) \Rightarrow \text{ future and present inputs}$$

Since output depends on past and future inputs the given system is **dynamic system**

10. Determine whether the following systems are static or dynamic

$\boldsymbol{y(n) = sinx(n)}$

$$y(0) = sinx(0) \quad \Rightarrow \text{ present input}$$

$$y(-1) = sinx(-1) \quad \Rightarrow \text{ present input}$$

$$y(1) = sinx(1) \quad \Rightarrow \text{ present input}$$

Since output depends on present input the given system is **Static system**

11. Determine whether the following systems are time invariant or not

$\boldsymbol{y(t) = x(t)sinwt}$

Output due to input delayed by T seconds

$$y(t,T) = x(t-T)sinwt$$

Output delayed by T seconds

$$y(t-T) = x(t-T)sinw(t-T)$$

$$\because y(t,T) \neq y(t-T)$$

The given system is **time variant**

12. Determine whether the following systems are time invariant or not

$\boldsymbol{y(n) = x(-n+2)}$

Output due to input delayed by k seconds

$$y(n,k) = x(-n+2-k)$$

Output delayed by k seconds

$$y(n-k) = x(-(n-k)+2) = x(-n+k+2)$$

$$\because \boldsymbol{y(n,k) \neq y(n-k)}$$

The given system is **time variant**

13. Determine whether the following systems are causal or not

$$\boldsymbol{y(t) = \frac{dx(t)}{dt} + 2x(t)}$$

The given equation is differential equation and the output depends on past input. Hence the given system is **Causal**

14. Determine whether the following systems are causal or not

$$\boldsymbol{y(n) = sinx(n)}$$

$$y(0) = sinx(0) \quad \Rightarrow \text{ present input}$$
$$y(-1) = sinx(-1) \quad \Rightarrow \text{ present input}$$
$$y(1) = sinx(1) \quad \Rightarrow \text{ present input}$$

Since output depends on present input the given system is **Causal system**

15. Determine whether the following systems are stable or not

$$\boldsymbol{h(t) = e^{-4t}u(t)}$$

$$\text{Condition for stability } \int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$$

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau = \int_{-\infty}^{\infty} |(e^{-4\tau})u(\tau)|\, d\tau = \int_{0}^{\infty} e^{-4\tau} d\tau = \left[\frac{e^{-4\tau}}{-4}\right]_0^{\infty} = \frac{1}{4}$$

$$\because \int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty \text{ the given system is } \textbf{stable}$$

16. Determine whether the following systems are stable or not

$$\boldsymbol{y(n) = 3x(n)}$$

$$\text{Let } x(n) = \delta(n), y(n) = h(n)$$
$$\Rightarrow h(n) = 3\delta(n)$$

$$\text{Condition for stability } \sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

$$\sum_{k=-\infty}^{\infty} |h(k)| = \sum_{k=0}^{\infty} |3\delta(k)| = \sum_{k=0}^{\infty} 3\delta(k) = 3$$

$$\because \delta(k) = 0 \; for \; k \neq 0 \quad and \quad \delta(k) = 1 \; for \; k = 0$$

$$\because \sum_{k=-\infty}^{\infty} |h(k)| < \infty \text{ the given system is } \textbf{stable}$$

Properties of Fourier Series

1) Linearity If $x(t) \xleftrightarrow{F.S} x(n)$ and $y(t) \xleftrightarrow{F.S} y(n)$ then

$$z(t) = a\,x(t) + b\,y(t) \xleftrightarrow{F.S} x(n) = a \cdot x(n) + b \cdot y(n)$$

Proof: from exponential f.s.c of $z(t)$ is

$$z(n) = \frac{1}{T}\int_0^T z(t) \cdot e^{-jnw_0 t} \cdot dt$$

$$= \frac{1}{T}\int_0^T (a\,x(t) + b\,(y(t))) \cdot e^{-jnw_0 t} \cdot dt$$

$$= \frac{1}{T}\left[a \underbrace{\int_0^T x(t) \cdot e^{-jnw_0 t} \cdot dt}_{x(n)} + b \int_0^T y(t) \cdot e^{-jnw_0 t} \cdot dt \right.$$

$$= \frac{a}{\cancel{T}} \cdot x(n) + \frac{b}{\cancel{T}} \cdot y(n)$$

$$\boxed{z(n) = a \cdot x(n) + b \cdot y(n)}$$

This property is used to analyse signals which are represented as linear combination of other signals.

2) Time Shifting or Translation

If $x(t) \longleftrightarrow x(n)$ then $z(t) = x(t - t_0) \longleftrightarrow z(n) = e^{-jnw_0 t_0} \cdot x(n)$

Fourier coefficients of $x(t - t_0)$ will be

$$z(n) = \frac{1}{T}\int_0^T x(t - t_0) \cdot e^{-jnw_0 t} \cdot dt$$

as $t \to 0$ $m \to$ ~~T~~ $t_0 = -t_0$

put $t - t_0 = m$ $\quad t = m + t_0$

$dt = dm$

$$\left.\begin{array}{ll} t = 0 & m = -t_0 \\ t \to T & m = T - t_0 \end{array}\right\} \text{Limits}$$

$$z(n) = \frac{1}{T}\int_{-t_0}^{T-t_0} x(m) \cdot e^{-jn\omega_0(m+t_0)} \cdot dm$$

$$= \frac{1}{T}\int_{-t_0}^{T-t_0} x(m) \cdot e^{-jn\omega_0 m}\, dm \cdot e^{-jn\omega_0 t_0}$$

$$\boxed{z(n) = x(n) \cdot e^{-jn\omega_0 t_0}}$$

3. Frequency Shift

If $x(t) \xleftrightarrow{FS} x(n)$ then

$$z(t) = e^{jm\omega_0 t} \cdot x(t) \xleftrightarrow{F.S} z(n) = x(n-m)$$

$$z(n) = \frac{1}{T}\int_0^T z(t) \cdot e^{-jm\omega_0 t} \cdot dt$$

$$= \frac{1}{T}\int_0^T e^{jm\omega_0 t} \cdot x(t) \cdot e^{-jn\omega_0 t} \cdot dt$$

$$= \frac{1}{T}\int_0^T x(t) \cdot e^{-j(n-m)\omega_0 t} \cdot dt$$

$$\boxed{Z(m) = X(n-m)}$$

4) Time scaling

If $x(t) \longleftrightarrow x(n)$ then $z(t) = x(at) \longleftrightarrow z(n) = x(n)$

An operation that in general changes the period of the underlying signal

$$x(n) = \frac{1}{T}\int_0^T x(t).\, e^{-jn\omega_0 t}.\, dt$$

Sol: Since $x(t)$ is periodic. then $z(t) = x(at)$ is also periodic.

If T is the period, then period of $z(t)$ will be T/a

Illy if frequency of $x(t)$ is ω_0. The frequency of $z(t) = x(at)$. will be $a\omega_0$, since 't' is multiplied by factor 'a'.

$$z(n) = \frac{1}{T/a}\int_0^{T/a} z(t).\, e^{-jna\omega_0 t}.\, dt$$

$$= \frac{1}{T/a}\int_0^{T/a} x(at).\, e^{-jna\omega_0 t}.\, dt$$

By putting $\tau = at$ which also yields $d\tau = at$

$t \to 0 \quad \tau \to 0$
$t \to T/a \quad \tau \to T$

$$z(n) = \frac{1}{T}\int_0^{\tau} x(\tau)\, e^{-jn\omega_0 \tau}.\, d\tau$$

$$\boxed{z(n) = x(n)}$$

5) Time Differentiation

If $x(t) \xleftrightarrow{F.S} x(k)$ then $\frac{d\,x(t)}{dt} \xleftrightarrow{F.S} jk\omega_0 x(k)$

$$x(t) = \sum_{n=-\infty}^{\infty} x(k)\, e^{jn\omega_0 t}$$

differentiating on both sides

$$\frac{d\,x(t)}{\partial t} = \sum_{k=-\infty}^{\infty} [jk\omega_0] . e^{jk\omega_0 t} . x(k)$$

$$\frac{d\,x(t)}{dt} = \sum_{n=-\infty}^{\infty} [jk\omega_0 x(n)] . e^{jk\omega_0 t}$$

$$\boxed{\frac{d\,x(t)}{dt} \longleftrightarrow jk\omega_0 x(k)}$$

$$\frac{d^k x(k)}{dt^k} \rightleftharpoons (jk\omega_0)^k X(k)$$

6) Convolution in time

If $x(t) \xleftrightarrow{F.S} X(k)$ and $y(t) \xleftrightarrow{F.S} Y(K)$ then $z(t) = x(t) * y(t)$

$$z(n) \xleftrightarrow{F.S} T X(n) Y(n)$$

we know that

$$z(k) = \frac{1}{T}\int_0^T z(n) . e^{-jn\omega_0 t} . dt$$

$$= \frac{1}{T}\int_{<T>} [x(t) * y(t)] e^{-jn\omega_0 t} . dt$$

we know that Convolution $\underbrace{x(t) * y(t) = \int x(\tau) y(t-\tau) . d\tau}_{\text{is called as periodic Convolution.}}$

By substituting

$$= \frac{1}{T}\int_{<T>}\int_{<T>} x(\tau) y(t-\tau) . d\tau \, e^{-jn\omega_0 t} . dt . d\tau$$

putting $t - \tau = m$ then $dt = dm$

$$= \frac{1}{T}\int_0^{T_0}\int_0^{T_0} x(\tau)\, y(t-\tau)\, d\tau\, e^{-jn\omega_0 t} \cdot \frac{e^{j\omega_0 n\tau}}{e^{jn\omega_0\tau}} \cdot dt$$

$$= \frac{1}{T}\int_{<T>}\int_{<T>} x(\tau) \cdot y(t-\tau) \cdot d\tau \cdot \frac{e^{-jn\omega_0(t-\tau)}}{e^{-jn\omega_0\tau}} \cdot dt$$

$$= \frac{1}{T}\int_{<T>}\int_{<T>} \left(x(\tau) \cdot e^{-jn\omega_0\tau} \cdot d\tau\right)\left(y(t-\tau) \cdot e^{-jn\omega_0(t-\tau)} \cdot dt\right)$$

$$= \underbrace{\left(\frac{1}{T}\int_{<T>} x(\tau) \cdot e^{-jn\omega_0\tau} \cdot d\tau\right)}_{X(k).}\ T. \underbrace{\left(\frac{1}{T}\int_{<T>} y(t-\tau) \cdot e^{-jn\omega_0(t-\tau)} \cdot dt\right)}_{\substack{\text{if } t-\tau = m \\ Y(k)}}$$

$$X(k). \quad T.$$

$$= T \cdot X(k) \cdot Y(k)$$

7) <u>Multiplication in time</u>

$$x(t) \rightleftharpoons X(k)$$

$$y(t) \rightleftharpoons Y(k)$$

$$x(t) \cdot y(t) \xrightleftharpoons{F.S} X(k) \circledast Y(k)$$

$$x(t) = \sum_{k=-\infty}^{\infty} X(k)\, e^{jk\omega_0 t}, \quad y(t) = \sum_{m=-\infty}^{\infty} Y(m) \cdot e^{jm\omega_0 t}$$

$$x(t) \cdot y(t) = \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} X(k).Y(k) \cdot e^{j(m+k)\omega_0 t}$$

$$m+k = n$$

$$m = n-k$$

$$= \sum_{n=-\infty}^{\infty} \boxed{\sum_{k=-\infty}^{\infty} X(k)\, Y(n-k)}\, e^{jn\omega_0 t}$$

8. ~~[illegible]~~ Integration in time

$$x(t) \rightleftharpoons X(k)$$

$$\int_{-\infty}^{t} x(\tau).d\tau \rightleftharpoons \frac{C_n}{jn\omega_o}$$

$$x(t) = \sum_{k=-\infty}^{\infty} X(k).e^{jk\omega_0 t}$$

$t \to \tau$

$$x(\tau) = \sum_{k=-\infty}^{\infty} X(k).e^{jk\omega_0 \tau}$$

integrating on both sides

$$\int_{-\infty}^{t} x(\tau)d\tau = \sum_{k=-\infty}^{\infty} \frac{X(k)}{jk\omega_0}.e^{jk\omega_0 \tau}$$

by comparing coefficient

$$\boxed{\int_{-\infty}^{t} x(\tau)d\tau = \frac{X(k)}{jk\omega_0}}$$

9. Parseval's Power Theorem

$$x(t) \rightleftharpoons C_n$$

$$P_{x(t)} = \sum_{n=-\infty}^{\infty} |C_n|^2$$

$z = a+ib$

$z^* = a-ib$

$|z| = \sqrt{a^2+b^2}$

$z.z^* = a^2+b^2$

Proof:

$$x(t) = \sum_{k=-\infty}^{\infty} C_k.e^{jn\omega_0 t}$$

Conjugate of $x(t)$ =

$$x^*(t) = \sum_{k=-\infty}^{\infty} C^*_{+k}.e^{-jn\omega_0 t}$$

$$x(t).x^*(t) = |x(t)|^2$$

$$P_{x(t)} = \frac{1}{T}\int_0^T |x(t)|^2 dt$$

$$= \frac{1}{T}\int_0^T x(t) \cdot x^*(t) \cdot dt$$

$$= \frac{1}{T}\int_0^T x(t) \sum_{k=-\infty}^{\infty} C_k^* \cdot e^{-jn\omega_0 t} \cdot dt$$

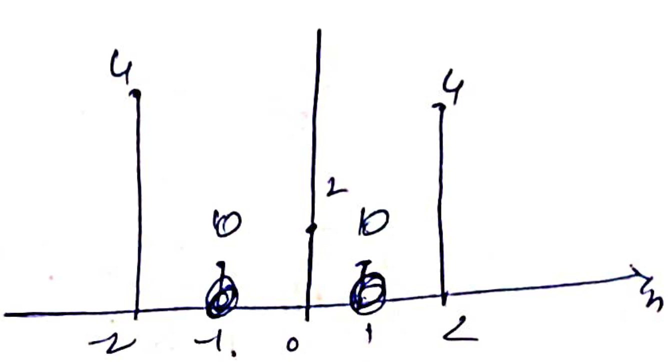
$$= \sum_{n=-\infty}^{\infty} C_n^* \cdot \frac{1}{T_0}\int_0^T x(t) \cdot e^{-jn\omega_0 t} \cdot dt$$

$$= \sum_{n=-\infty}^{\infty} C_n^* \cdot C_n$$

$$\boxed{P_{x(t)} = \sum_{n=-\infty}^{\infty} |C_n^*|^2}$$

Eg: Find the avg. power of signal $x(t)$, when C_n is given as

$$P_{x(t)} = \sum_{n=-\infty}^{\infty} |X(k)|^2$$



Method 1

$-\infty$ to -3

$X(k)=0$

$X(-2)=4 \quad X(-1)=0 \quad X(0)=2 \quad X(1)=0 \quad X(2)=4$

$$P_{x(t)} = |X(-2)|^2 + |X(0)|^2 + |X(2)|^2$$

$$= |4|^2 + |2|^2 + |4|^2$$

$$= 16 + 4 + 16 = 36 \text{ watts}$$

Method-II

$$x(t) = \sum_{k=-\infty}^{\infty} X(k)\, e^{jk\omega_0 t}$$

$$= X(-2)\cdot e^{-j2\omega_0 t} + X(0) + X(2)\, e^{j2\omega_0 t}$$

$$= 4e^{-j2\omega_0 t} + 2 + 4e^{j2\omega_0 t}$$

$$= 2 + 4\left(e^{-j2\omega_0 t} + e^{j2\omega_0 t}\right)$$

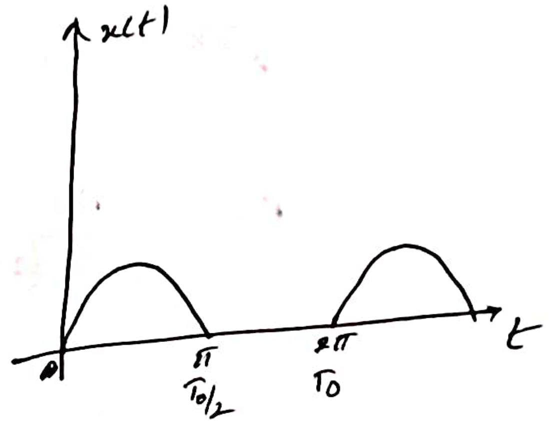
$x(t) = 2 + 4.2\cos 2\omega_0 t$ Power of $A\cos\omega_0 t$

$= 2 + 8\cos 2\omega_0 t$ $\frac{A_0^2}{2}$ for cosine signals

$$P_{avg\,x(t)} = |2|^2 + \frac{(8)^2}{2}$$

$$= 4 + 32 = 36 \text{ watts}$$

1. Find the exponential fourier series and plot magnitude & phase spectrum of half wave rectified sine wave.



Sol:

$$x(t) = \begin{cases} A \sin \omega_0 t & \text{for } 0 \le t \le T_0/2 \\ 0 & \text{for } T_0/2 \le t < T_0 \end{cases}$$

$T_0 = 2\pi = T$

$$\omega_0 = \frac{2\pi}{T} = \cancel{\frac{2\pi}{2\pi}} = 1$$

$$X(k) = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} A \sin \omega_0 t \cdot e^{-jk\omega_0 t}\, dt$$

$$= \frac{A}{2\pi}\int_0^{\pi} \sin \omega_0 t \cdot e^{-jk\omega_0 t}\, dt = \frac{A}{2\pi}\int_0^{\pi} \sin t \cdot e^{-jkt}\, dt$$

$$\int e^{ax} \cdot \sin(bx+c)\, dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin(bx+c) - b\cos(bx+c)\right]$$

with $a = -jk$, $b = 1$, $c = 0$ and $x = t$

$$X(k) = \frac{A}{2\pi}\left[\frac{e^{-jkt}}{(-jk)^2+1}\left[-jk \sin(t) - \cos t\right]\right]_0^{\pi}$$

$$= \frac{A}{2\pi}\left\{\frac{e^{-j\pi k}}{(-jk)^2+1}\left[-jk \sin \pi - \cos \pi\right] - \frac{e^0}{(-jk)^2+1}\left[-jk \sin 0 - \cos 0\right]\right.$$

$$= \frac{A}{2\pi}\left[\frac{e^{-j\pi k}}{(-jk)^2+1} + \frac{1}{(-jk)^2+1}\right] = \frac{A}{2\pi(-k^2+1)}\left\{e^{-j\pi k}+1\right\}$$

$$e^{-j\pi k} = (-1)^k$$

$$X(k) = \frac{A}{2\pi(1-k^2)}\left[(-1)^k+1\right] \quad \text{for } k \neq \pm 1$$

$$= \frac{A}{2\pi(1-k^2)} \quad \text{for } k = 0, \pm 2, \pm 4, \pm 6$$

$$= 0 \quad \text{for } k = \pm 3, \pm 5, \pm 6$$

putting for $k=1$

$$X(k) = \frac{A}{2\pi}\int_0^{\pi} \sin t \cdot e^{jt} \cdot dt$$

$$= \frac{A}{2\pi}\int_0^{\pi} \frac{e^{jk} - e^{-jk}}{2j} \cdot e^{-jt} \, dt$$

$$= \frac{A}{2\pi}\int_0^{\pi} \frac{(1-e^{-j2t})}{2j} dt = \frac{A}{4\pi}\int_0^{\pi} (1-e^{-j2t}) \, dt$$

$$= \frac{A}{j4\pi}\left[[t]_0^{\pi} - \frac{1}{2j}\left[e^{-j2t}\right]_0^{\pi}\right]$$

$$= \frac{A}{j4\pi}\left[\pi - 0 + \frac{1}{j2}\left[e^{-j2\pi} - e^{0}\right]\right]$$

$$= \frac{A}{j4}$$

②

if $k = -1$

$$X(k) = \frac{A}{2\pi}\int_0^{\pi} \sin t \cdot e^{jt} \cdot dt = \frac{A}{2\pi}\int_0^{\pi} \frac{e^{jt} - e^{-jt}}{2j} e^{jt} \cdot dt$$

$$= \frac{A}{j4\pi}\int_0^{\pi} (e^{2jt} - 1)\, dt = \frac{A}{4\pi j}\left[\frac{1}{2j}\left[e^{j2t}\right]_0^{\pi} - \left[t\right]_0^{\pi}\right]$$

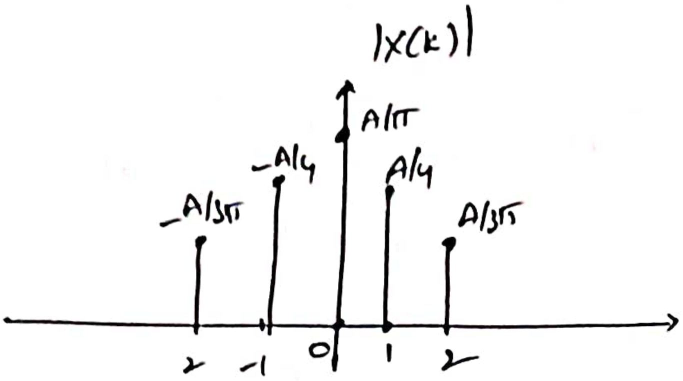
$$= \frac{A}{4\pi j}\left[\frac{1}{2j}\left[e^{j2\pi} - e^{0}\right] - \left[\pi - 0\right]\right]$$

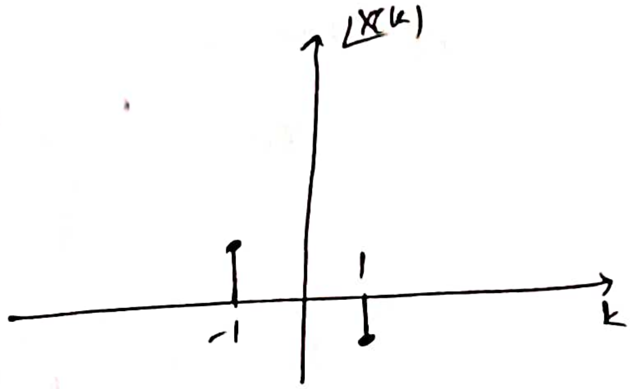
$$= \frac{-A}{4j}$$

$$X(k) = \begin{cases} \frac{A}{\pi(1-k^2)} & \text{for } k = 0, \pm 2, \pm 4 \ldots \\ 0 & \text{for } k = \pm 3, \pm 5, \ldots \\ jA/4 & \text{for } k = -1 \\ -jA/4 & \text{for } k = 1 \end{cases}$$

$$|X(k)| = \begin{cases} \left|\frac{A}{\pi(1-k^2)}\right| & k = 0, \pm 2, \pm 4, \\ 0 & k = \pm 3, \pm 5, \ldots \\ \frac{A}{4} & k = \pm 1 \end{cases}$$

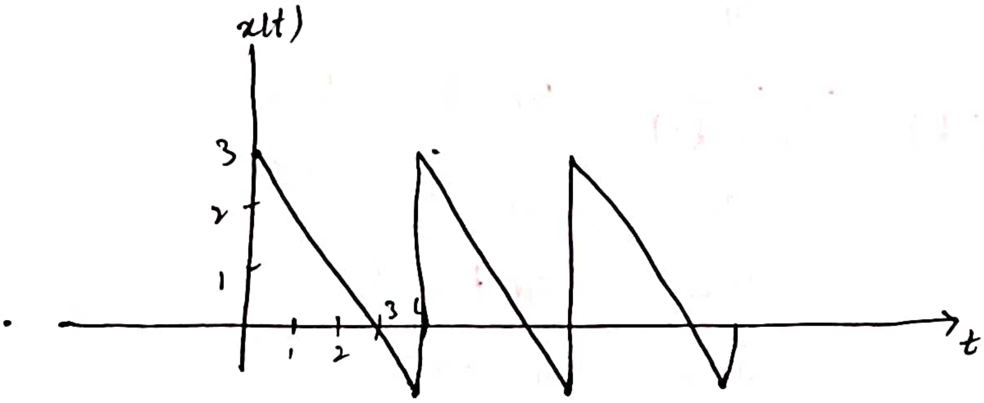
$$\angle X(k) = \begin{cases} \tan^{-1}\left(\frac{-A/4}{0}\right) = -\pi/2 & \text{for } k = 1 \\ \tan^{-1}\left(\frac{A/4}{0}\right) = \pi/2 & \text{for } k = -1 \\ 0 & \text{for } k \neq \pm 1 \end{cases}$$





2. A periodic signal with a period of 4 sec is described over one fundamental period by $x(t) = 3-t$ $0 \le t \le 4$. plot the signal & find the exponential fourier series. plot the amplitude & Phase spectrum.

Sol:



$x(t) = 3-t \qquad 0 \le t \le 4$

$\omega_0 = \frac{2\pi}{4} = \pi/2$

$$X(k) = \frac{1}{T}\int_0^T x(t).\, e^{-jk\omega_0 t}.\, dt$$

$$= \frac{1}{4}\int_0^4 (3-t)\, e^{-jk\omega_0 t}.\, dt$$

$$= \frac{1}{4}\int_0^4 3.e^{-jk\omega_0 t}.\, dt - \frac{1}{4}\int_0^4 t\, e^{-jk\omega_0 t}.\, dt$$

$$= \frac{3}{4}\left[\frac{-1}{jk\omega_0}\left[e^{-jk\omega_0 t}\right]_0^4\right] - \frac{1}{4}\left[\frac{-t}{jk\omega_0} e^{-jk\omega_0 t}\right]_0^4 - \frac{1}{j}$$

Question: Find $X(k)$ C_n interms of $X'(k)$ C_n', where $x(t) = C_n$, $y(t) = C_n'$

1) $y(t) = x(t+1) + x(t-1)$

2) $y(t) = e^{-j2\omega_0 t} \cdot x(t)$

3) $y(t) = \frac{d^2}{dt^2} x(t)$

4) $y(t) = \text{odd}[x(t)]$

5) $y(t) = \text{Real}[x(t)]$

Sol: 1. $x(t) \rightleftharpoons C_n$

$$x(t \pm t_0) \rightleftharpoons C_n \cdot e^{\pm jn\omega_0 t}$$

$$x(t+1) = C_n \cdot e^{jn\omega_0 t}$$

$$x(t-1) = C_n \cdot e^{-jn\omega_0}$$

$$x(t+1) + x(t-1) = C_n \cdot e^{jn\omega_0} + C_n e^{-jn\omega_0}$$

$$= C_n (e^{jn\omega_0} + e^{-jn\omega_0})$$

$$= 2\cos n\omega_0 \cdot C_n$$

$$= C_n'$$

2. $$e^{jm\omega_0 t} \cdot x(t) \rightleftharpoons C_{n-m}$$

$$y(t) = e^{-j2\omega_0 t} \cdot x(t) \rightleftharpoons C_n'$$

$$e^{+j(-2)\omega_0 t} \cdot x(t) \rightleftharpoons C_n' = C_{n-m}$$

m

$$= C_{n+2}$$

3. $$\frac{d^k}{dt^k} x(t) \rightleftharpoons (jn\omega_0)^k X(k)$$

$$\frac{d^2}{dt^2} x(t) = j^2 n^2 \omega_0^2 C_n$$

$$= -n^2 \omega_0^2 C_n$$

4. $y(t) = \text{odd}[x(t)]$

$$= \frac{x(t) - x(-t)}{2}$$

$$\hat{y(t)} = \frac{C_n - C_{-n}}{2} = C_n'$$

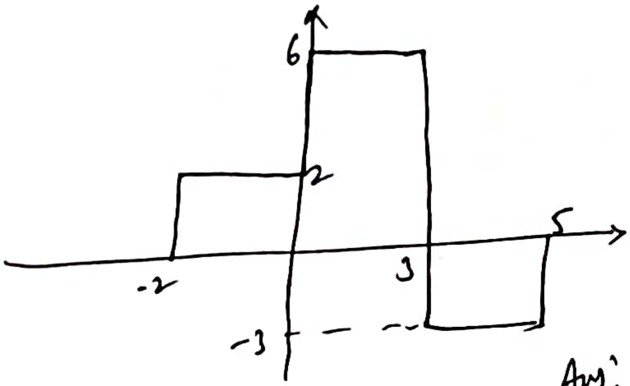
5. $y(t) = \text{Real}[x(t)]$

$$= \frac{x(t) + x^*(t)}{2}$$

$$= \frac{C_n + C_{-n}^*}{2}$$

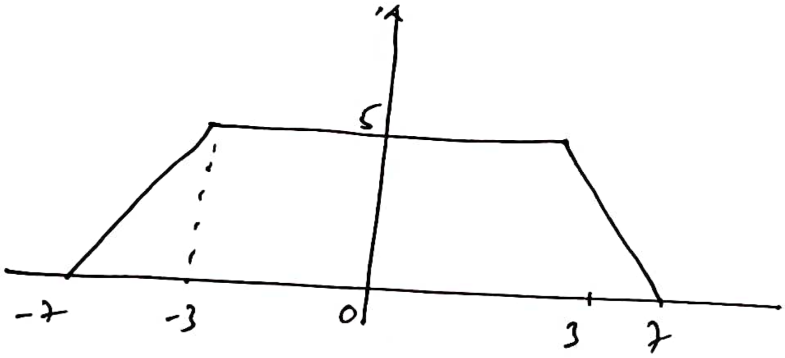
$$C_n' = \frac{C_n + C_{-n}^*}{2}$$

1. Represent the following signal using shifted unit step signal

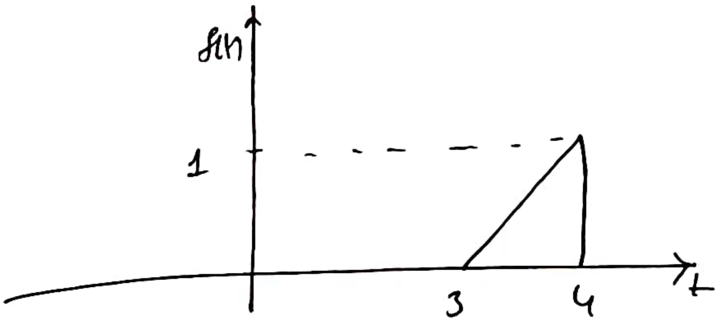


Ans: $2u(t+2) + 4u(t) - 9u(t-3) + 3u(t-5)$

2. Represent the following signal using shifted ramp signal and shifted unit step signals



Ans: $\frac{5}{4}r(t+7) - \frac{5}{4}r(t+3) - \frac{5}{4}r(t-3) + \frac{5}{4}r(t-7)$



$f(t) = r(t-3) - u(t-4) - r(t-4)$

Fourier Transform

Let a periodic signal $x(t)$ with period T. The Complex fourier series representation of $x(t)$ is given as

$$x(t) = \sum_{k=-\infty}^{\infty} X(k) \cdot e^{jk\omega_0 t}$$

where $\frac{2\pi}{T}$ is angular frequency ω_0.

$$\Delta f = \frac{1}{T_0}$$

$$x(t) = \sum_{k=-\infty}^{\infty} X(k) \cdot e^{jk\frac{2\pi}{T} \times t} = \sum_{k=-\infty}^{\infty} X(k) \cdot e^{jk2\pi \Delta f t}$$

$$X(k) = \frac{1}{T_0} \int_{0}^{T} x(t) \cdot e^{-jk\omega_0 t} \cdot dt$$

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} \int_{0}^{T} x(t) \cdot e^{-jk\omega_0 t} \cdot dt \cdot e^{jk2\pi f t}$$

$T = T/2$

$$= \sum_{k=-\infty}^{\infty} \Delta f \int_{0}^{T} x(t) \cdot e^{-jk2\pi \Delta f t} \cdot dt \cdot e^{jk2\pi \Delta f t}$$

$T_0/2$

$$= \sum_{k=-\infty}^{\infty} \left[\Delta f \int_{-T/2}^{T/2} x(t) \cdot e^{-j2\pi k \Delta f t} \cdot dt \right] \cdot e^{jk2\pi \Delta f t}$$

When $T \to \infty$ Δf becomes df. & $k\Delta f = f$.

$$= \int_{-\infty}^{\infty} \int_{-T/2}^{T/2} x(t) \cdot e^{-j2\pi \Delta f k \cdot t} \cdot dt \cdot e^{jk2\pi \Delta f t} \cdot df$$

$$x(t) = \int_{-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi ft} \cdot dt}\, \cdot e^{j2\pi ft} \cdot df$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} \cdot d\omega$$

$$\boxed{\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt \\ x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} \cdot d\omega \end{aligned}} \rightarrow \text{F.T pair}$$

Fourier Transform of single sided exponential Signal.

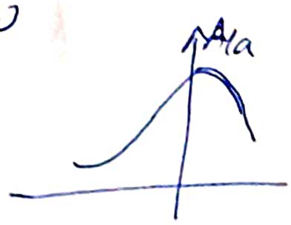
$$x(t) = Ae^{-at} \; ; \text{ for } t \geq 0$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt$$

$$= \int_{-\infty}^{\infty} Ae^{-at} \cdot e^{-j\omega t} \cdot dt = \int_{0}^{\infty} Ae^{-(a+j\omega)t} \cdot dt$$

$$= \left[\frac{Ae^{-(a+j\omega)t}}{-(a+j\omega)} \right]_{0}^{\infty} = \frac{A}{a+j\omega}$$

$$|X(\omega)| = \frac{A}{\sqrt{a^2+\omega^2}}$$



Fourier Transform of two-sided exponential s/l.

$$x(t) = Ae^{-a|t|} \; ; \; \forall t$$

$$x(t) = Ae^{-at}, \quad t = 0 \text{ to } \infty$$

$$Ae^{at}, \quad t = -\infty \text{ to } 0$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt$$

$$= \int_{-\infty}^{0} Ae^{at} \cdot e^{-j\omega t} \cdot dt + \int_{0}^{\infty} Ae^{-at} \cdot e^{-j\omega t} \cdot dt$$

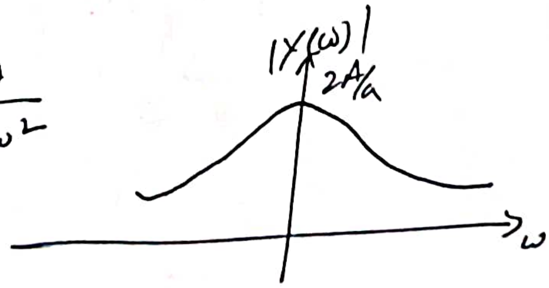
$$= \int_{-\infty}^{0} Ae^{(a-j\omega)t} \cdot dt + \int_{0}^{\infty} Ae^{-(a+j\omega)t} \cdot dt$$

$$= \left[\frac{A \cdot e^{(a-j\omega)t}}{a-j\omega} \right]_{-\infty}^{0} + \left[\frac{A \cdot e^{-(a+j\omega)t}}{-(a+j\omega)} \right]_{0}^{\infty}$$

$$= \frac{Ae^{0}}{a-j\omega} - \frac{Ae^{-\infty}}{a-j\omega} + \frac{Ae^{-\infty}}{-(a+j\omega)} - \frac{Ae^{0}}{-(a+j\omega)}$$

$$= \frac{A}{a-j\omega} + \frac{A}{a+j\omega} = \frac{2aA}{a^2+\omega^2}$$

$$|X(\omega)| = \frac{2aA}{a^2+\omega^2}$$



Properties of Fourier Transform

1. Linearity : If $x(t) \xleftrightarrow{F.T} X(\omega)$ & $y(t) \xleftrightarrow{F.T} Y(\omega)$.

then $a.x(t) + b.y(t) \longleftrightarrow aX(\omega) + bY(\omega)$

$$Z(\omega) = \int_{-\infty}^{\infty} z(t).e^{-j\omega t}.dt$$

$$= \int_{-\infty}^{\infty} [ax(t) + by(t)]e^{-j\omega t}.dt$$

$$= a\int_{-\infty}^{\infty} x(t)e^{-j\omega t}.dt + b\int_{-\infty}^{\infty} y(t).e^{-j\omega t}.dt$$

$$= aX(\omega) + bY(\omega)$$

2 Time Shifting :

If $x(t) \longleftrightarrow X(\omega)$

$x(t-t_0) \longleftrightarrow e^{-j\omega t_0}.X(\omega)$

$$Z(\omega) = \int_{-\infty}^{\infty} z(t).e^{-j\omega t}.dt$$

$$= \int_{-\infty}^{\infty} x(t-t_0).e^{-j\omega t}.dt$$

let $t - t_0 = \tau$

$t = \tau + t_0$

$$= \int_{-\infty}^{\infty} x(\tau).e^{-j\omega \tau}.e^{-j\omega t_0}.d\tau$$

$$= e^{-j\omega t_0}.X(\omega)$$

3 Frequency shifting :→ If $x(t) \xleftrightarrow{FT} X(\omega)$ then $y(t) = e^{j\omega_0 t} \cdot x(t)$

$$Y(\omega) = X(\omega - \omega_0)$$

$$Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} \cdot dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot e^{-j\omega_0 t} \cdot dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot e^{-jt(\omega - \omega_0)} \cdot dt = \int_{-\infty}^{\infty} x(t) \cdot e^{-jt(\omega - \omega_0)} \cdot dt$$

$$= X(\omega - \omega_0)$$

4 Time Scaling :→

If $x(t) \xleftrightarrow{FT} X(\omega)$

$$y(t) = x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

$$Y(\omega) = \int_{-\infty}^{\infty} x(at) \cdot e^{-j\omega t} \cdot dt$$

$at = \tau \qquad t = \tau/a$

$$dt = \frac{1}{a} d\tau$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(\tau) e^{-j\left(\frac{\omega}{a}\right)\tau} \cdot d\tau$$

$$= 1/|a| \cdot X\left(\frac{\omega}{a}\right)$$

5 **Frequency Differentiation:**

If $x(t) \xleftrightarrow{F.T} X(\omega)$ then $-jt.x(t) \longleftrightarrow \frac{d X(\omega)}{d\omega}$

Meaning: Differentiating the frequency spectrum is equalent to multiplying the time domain s/e by complex number $-jt$.

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}.dt$$

$$\frac{d\, X(\omega)}{d\omega} = \int_{-\infty}^{\infty} x(t). \frac{d}{d\omega}\left[e^{-j\omega t}\right].dt$$

$$= -jt.\int_{-\infty}^{\infty} x(t).e^{-j\omega t}.dt$$

$$= -j\int_{-\infty}^{\infty} t.x(t).e^{-j\omega t}.dt$$

6 **Time Differentiation:**

If $x(t) \longleftrightarrow X(\omega)$ then $\frac{d\, x(t)}{dt} \longleftrightarrow j\omega X(\omega)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega).e^{j\omega t}.d\omega$$

$$\frac{d}{dt}x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega).\frac{de^{j\omega t}}{dt}.d\omega$$

$$= j\omega\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega).e^{j\omega t}.d\omega\right. = j\omega X(\omega)$$

$$\boxed{\frac{d^n}{dt^n}x(t) = (j\omega)^n X(\omega)}$$

7 Time Integration: → If $x(t) \xrightarrow{F.T} X(\omega)$

$$\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right] \xleftrightarrow{F.T} \frac{1}{j\omega} X(\omega)$$

$$x(t) = \frac{d}{dt}\int_{-\infty}^{t} x(\tau).\,d\tau$$

$$F[x(t)] = F\left[\frac{d}{dt}\int_{-\infty}^{t} x(\tau)\,d\tau\right]$$

By differentiating property. for right hand sid

$$= j\omega\, F\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right]$$

$$X(\omega) = j\omega\, F\left[\int_{-\infty}^{\infty} x(\tau)\,d\tau\right]$$

$$F\left[\int_{-\infty}^{\infty} x(\tau).\,d\tau\right] = \frac{1}{j\omega}.\,X(\omega)$$

8 Multiplication

If $x(t) \longleftrightarrow X(\omega)$ and $y(t) \longleftrightarrow Y(\omega)$ then

$$z(t) - x(t)\,.\,y(t) \longleftrightarrow Z(\omega)\ \frac{1}{2\pi}\left[X(\omega) * Y(\omega)\right]$$

$$Z(\omega) = \int_{-\infty}^{\infty} z(t)\, e^{-j\omega t}.\,dt$$

$$= \int_{-\infty}^{\infty} x(t)\, y(t).\, e^{-j\omega t}.\,dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega).\, e^{j\omega t}.\,dt$$

Putting for $x(t)$

$$z(\omega) = \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda).e^{j\lambda t}\, d\lambda\right].y(t)e^{-j\omega t}.dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)\int_{-\infty}^{\infty} y(t).e^{-j(\omega-\lambda)t}.dt.d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)\, Y(\omega-\lambda).d\lambda$$

$$= \frac{1}{2\pi}\left[X(\omega) * Y(\omega)\right]$$

Parseval's Energy theorem:→

$$x(t) \rightleftharpoons X(j\omega)$$

$$E_{x(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(j\omega)|^2 d\omega$$

Proof: $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega).e^{j\omega t}.d\omega$$

Conjugate of above equation

$$x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega).e^{-j\omega t}.d\omega$$

$$E_{x(t)} = \int_{-\infty}^{\infty}|x(t)|^2 dt \qquad |x(t)|^2 = x(t).x^*(t)$$

$$= \int_{-\infty}^{\infty} x(t).x^*(t)dt$$

$$= \int_{-\infty}^{\infty} x(t)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega).e^{-j\omega t}.d\omega\right\}dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) \cdot e^{-j\omega t} \cdot d\omega \right\} dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) \underbrace{\int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt}_{\Downarrow} \cdot d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) \cdot X(j\omega) \cdot d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega.$$

Duality

$$x(t) \xrightleftharpoons{F.W} X(j\omega)$$

$$X(t) \rightleftharpoons 2\pi x(-\omega)$$

Proof:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} \cdot d\omega$$

$$t = -t$$

$$x(-t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{-j\omega t} \cdot d\omega$$

$$2\pi \, x(-t) = \int_{-\infty}^{\infty} X(\omega) \cdot e^{-j\omega t} \cdot d\omega$$

$$t = \omega \ \& \ d\omega = dt.$$

$$2\pi \, x(-\omega) = \int_{-\infty}^{\infty} X(t) \cdot e^{-j\omega t} \cdot dt$$

$$=$$

Fourier Transform applied absolutely integrable signals. but dc values are not absolutely integrable. but we can apply.

$$\int_{-\infty}^{\infty} |x(t)|\,dt < \infty$$

$$x(t) = A_0$$

$$\int_{-\infty}^{\infty} A_0\,dt = A_0 \int_{-\infty}^{\infty} dt$$

$$= A_0 (t]_{-\infty}^{\infty}$$

$$= A_0 \{\infty - (-\infty)\}$$

$$= A_0 * \infty$$

$$= \infty$$

but $x(t) \rightleftharpoons X(j\omega) / X(\omega)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt$$

$$= \int_{-\infty}^{\infty} A_0 \cdot e^{-j\omega t} \cdot dt$$

$$= A_0 \left[\frac{e^{-j\omega t}}{-j\omega} \right]_{-\infty}^{\infty}$$

$$= \frac{A_0}{-j\omega} \left[e^{-j\omega \infty} - e^{-j\omega(-\infty)} \right]$$

$$= \frac{A_0}{\omega} \left(\frac{e^{j\infty} - e^{-j\infty}}{j} \right)$$

$$= \frac{A_0}{\omega} 2 \sin(\infty)$$

So here $\sin\infty$ not defined

$$x(t) \rightleftharpoons X(\omega) = A_0 \delta(\omega)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} \cdot d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} A_0 \delta(\omega) \cdot e^{j\omega t} \cdot d\omega$$

$$= \frac{A_0}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - 0) \cdot e^{j\omega t} \cdot d\omega$$

$$= \frac{A_0}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - 0) \cdot e^{0} \cdot d\omega$$

$$= \frac{A_0}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) \cdot d\omega = \underbrace{1}_{x(t)} = \frac{A_0}{2\pi}$$

$$\frac{A_0}{2\pi} \rightleftharpoons A_0 \delta(\omega)$$

$$A_0 \rightleftharpoons 2\pi A_0 \delta(\omega)$$

Fourier Transform of Impulse Signal

$$x(t) = \delta(t) \rightleftharpoons X(\omega)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt$$

$$= \int_{-\infty}^{\infty} \delta(t) \cdot e^{-j\omega t} \cdot dt$$

$$= \int_{-\infty}^{\infty} \delta(t-\underset{t_0}{\underbrace{0}}) \cdot e^{-j\omega t} \cdot dt$$

from Properties of Impulse Signal

$$x(t) \cdot \delta(t-t_0) = x(t_0) \cdot \delta(t-t_0)$$

$$= \int_{-\infty}^{\infty} \delta(t) \cdot e^{-j\omega \cdot 0} \cdot dt$$

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t) \cdot dt = 1$$

$$\boxed{X(\omega) = 1}$$

Fourier Transform of Exponential Signals

$$x(t) = e^{-at} u(t), \quad a > 0$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt$$

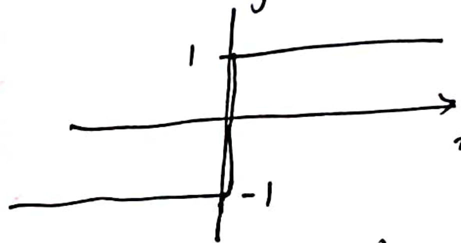
$$= \int_{-\infty}^{\infty} e^{-at} \cdot u(t) \cdot e^{-j\omega t} \cdot dt = \int_{0}^{\infty} e^{-at} \cdot e^{-j\omega t} \cdot dt = \int_{0}^{\infty} e^{-(a+j\omega)t} \cdot dt$$

$$= \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^{\infty} = \frac{1}{a+j\omega}$$

Fourier transform of signum function

$$x(t) = Sgn(t) = \begin{cases} -1, & t<0 \\ 0, & t=0 \\ 1, & t>0 \end{cases} \rightleftharpoons X(\omega) =$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t).\, e^{-j\omega t} \cdot dt$$



$$= \int_{-\infty}^{\infty} sgn(t)\, e^{-j\omega t} \cdot dt$$

$u(t) - u(-t)$

$$= \int_{-\infty}^{0} -1. e^{-j\omega t} \cdot dt + \int_{0}^{\infty} 1. e^{-j\omega t} \cdot dt$$

$$= \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{0}^{\infty} = \frac{1}{j\omega}\left[e^{-0} - e^{-\infty}\right] - \frac{1}{j\omega}\left[e^{-\infty} - e^{0}\right]$$

$$= \frac{1}{j\omega}\, 1-0 - \frac{1}{j\omega}\left[0-1\right] = \frac{2}{j\omega}$$

Fourier Transform of step s/l:

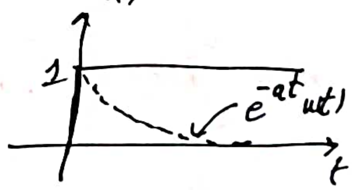
$$x(t),\ u(t) = \begin{matrix} 0 & t<0 \\ 1 & t>0 \end{matrix}$$

$X(\omega) =$ ___

step s/l is not absolutely integrable

$u(t)$ not converges

$u(t)$ has to makes converges



$e^{-at} \rightarrow$ is a limiting case.

So. we use signum function

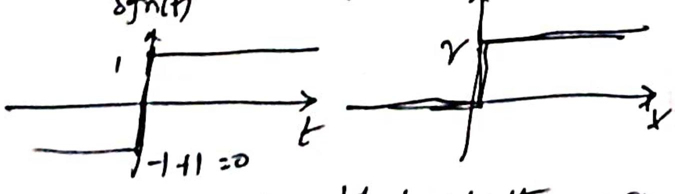
$sgn \rightleftharpoons 2/j\omega$

$$u(t) = \lim_{a \to 0} e^{-at} . u(t)$$

↓

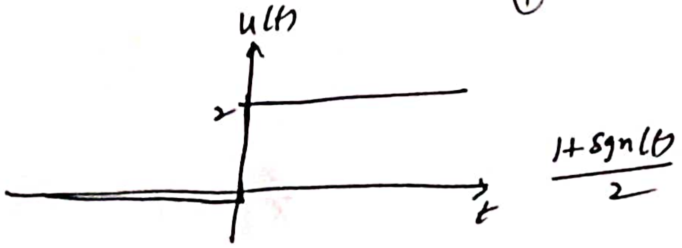
To this we need to apply F.T,

but this is very lengthy



amplitude shifting $1+Sgn(t)$

$u(t) = \frac{1+sgn(t)}{2}$



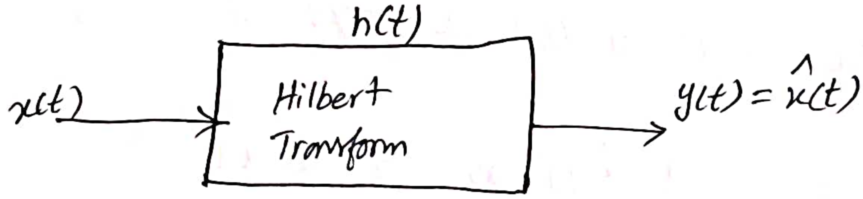
$F.T\{u(t)\} = \frac{1}{2} + \frac{1}{2}\, sgn(t)$

$X(\omega) = \cancel{2}\pi * \frac{1}{\cancel{2}}.\delta(\omega) + \frac{\cancel{2}}{j\omega} \times \frac{1}{\cancel{2}}$

$= \pi.\delta(\omega) + \frac{1}{j\omega}$

Hilbert Transform:

* It does not involve in change of domain.
* Hilbert transform of any signal is not an equivalent representation of the signal rather it is entirely different signal.
* Hilbert transform is a $\pm 90°$ phase shift which doesn't effect its amplitude
* If we multiply the spectrum of $x(t)$ by $-j$ and the spectrum of $x(t)$ by $+j$. we get hilbert transform of $F.T\{x(t)\}$.
*



$\therefore y(t) = x(t) * h(t)$

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau).d\tau}$$

taking the Fourier transform

$Y(\omega) = X(\omega) \circledast H(\omega) = X(\omega).H(\omega)$

convoluted.

$$F.T\left[\hat{x}(t)\right] = X(\omega).\, H(\omega)$$

$$\hat{x}(t) = IFT\{X(\omega).\, H(\omega)\}$$

$$H(f) = -j\,\mathrm{sgn}(f)$$

$$h(t) = \frac{1}{\pi t}$$

$$\mathrm{sgn}(t) \xleftrightarrow{F.T} \frac{2}{j\omega} = \frac{1}{j\pi f}$$

by duality property

$$\frac{1}{j\pi t} \longleftrightarrow \mathrm{sgn}(f)$$

$$\frac{1}{\pi t} \longleftrightarrow -j\,\mathrm{sgn}(f)$$

$$\boxed{y(t) = \hat{x}(t) = \int_{-\infty}^{\infty} \frac{x(\tau)}{\pi(t-\tau)}\, d\tau}$$

Property of Hilbert Transform

1) H.T of an odd signal is even and H.T of an even signal is odd.

2) If $\hat{x}(t)$ is H.T of $x(t)$ then H.T of $\hat{x}(t)$ is $-x(t)$.

3) $x(t)$ and $\hat{x}(t)$ are orthogonal

$$\int_{-\infty}^{\infty} x(t).\, \hat{x}(t).\, dt = 0$$

4) Energy contained in any signal $x(t)$ and energy in $\hat{x}(t)$ are same.

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt =$$

both amplitudes are same so Energy is equal

⑥

Σ H.T of derivation of any signal is equal to derivative of Hilbert transform of that signal.

$$H.T\left[\frac{dx(t)}{dt}\right] = \frac{d}{dt}\left[H.T\{x(t)\}\right]$$

Sampling Theorem:

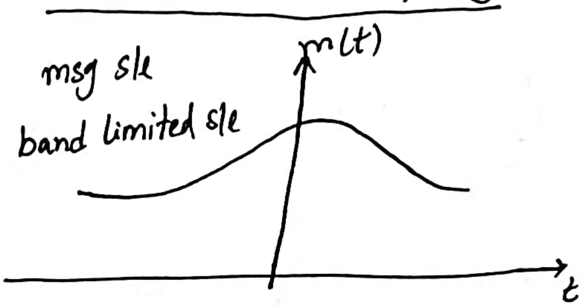
A signal can be represented in its samples and can be recovered back when sampling frequency is greater than or equal to twice of maximum frequency component present in the signal.

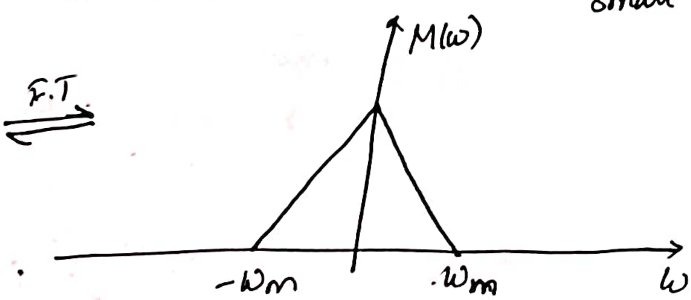
Condition for sampling

Band limited s/e F.T is nonzero for small range of freq.

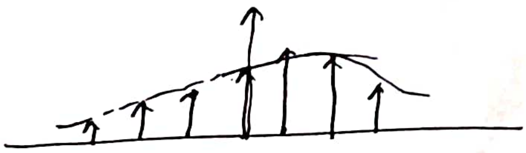
msg s/e

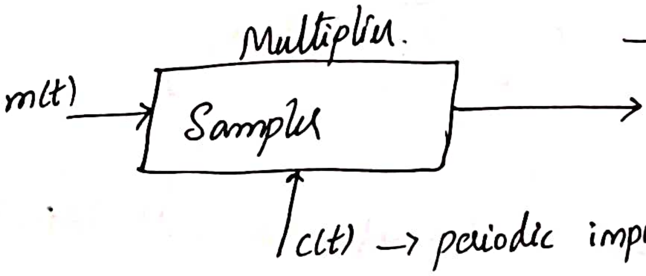
band limited s/e





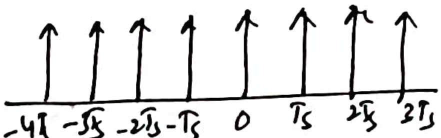
ω_m = max. frequency component of m(t).





$s(t) = m(t).c(t)$

c(t) → periodic impulse train



$$c(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_s)$$

T_s → Sampling period

$\omega_s = \frac{2\pi}{T_s}$ = Sampling frequency.

$$s(t) \xrightleftharpoons{FT} S(\omega)$$

$$m(t) \cdot c(t) \rightleftharpoons \frac{1}{2\pi}\left[M(\omega) * C(\omega)\right]$$

$$S(\omega) = \frac{1}{2\pi}\left[M(\omega) * C(\omega)\right]$$

$$= \frac{1}{2\pi}\left[M(\omega) * \omega_s \sum_{n=-\infty}^{\infty} \delta(\omega - \omega_s \cdot n)\right]$$

$$= \frac{1}{2\pi}\left[M(\omega) * \omega_s \sum_{n=-\infty}^{\infty} \delta(\omega - n \cdot \omega_s)\right]$$

$$= \frac{\omega_s}{2\pi}\left[M(\omega) * \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)\right]$$

Rearranging

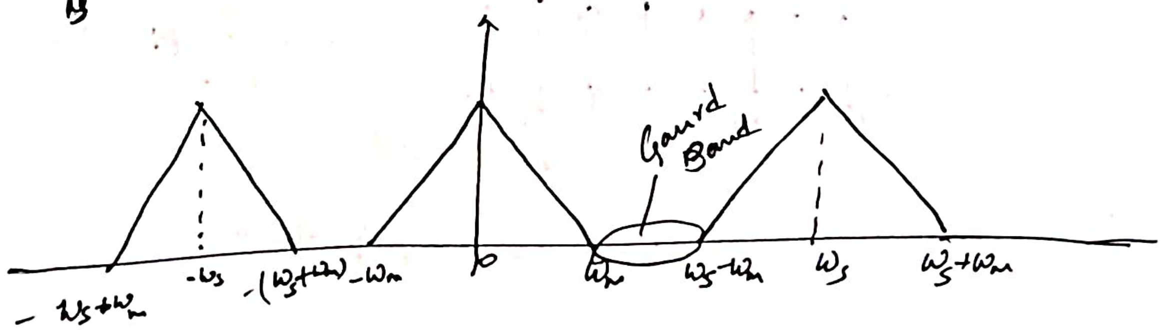
$$= \frac{\omega_s}{2\pi}\left[M(\omega) * \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)\right]$$

$$= \frac{\omega_s}{2\pi}\left[\sum_{n=-\infty}^{\infty} M(\omega) * \delta(\omega - n\omega_s)\right]$$

By using property $x(t) * \delta(t - t_1) = x(t - t_1)$

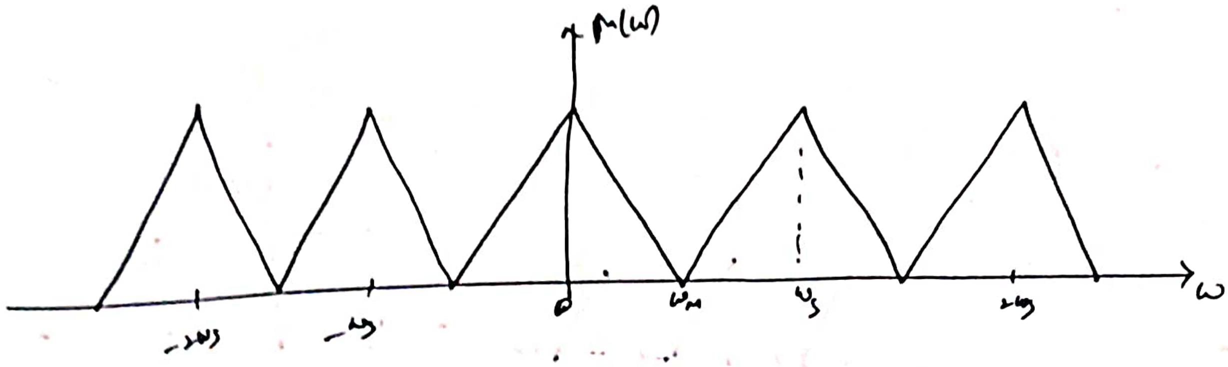
$$= \frac{1}{T_s}\left[\sum_{n=-\infty}^{\infty} M(\omega - n\omega_s)\right]$$

$$S(\omega) = \frac{1}{T_s}\left[\cdots M(\omega + n\omega_s) + M(\omega) + M(\omega - \omega_s) + \cdots \right]$$

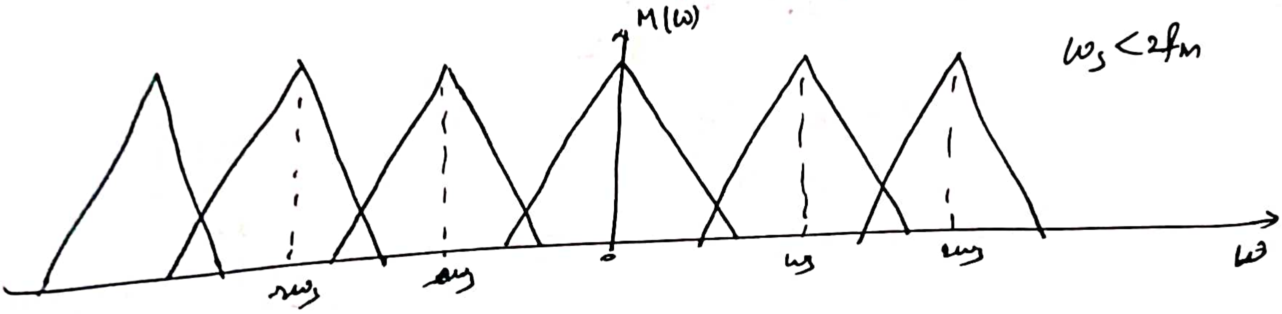


$\omega_s - \omega_m > \omega_m$, $\underline{\omega_s > 2\omega_m}$ $\boxed{\omega_s > 2\omega_m}$

$\omega_s = 2.\omega_m$



$\omega_s < 2f_m$



Nyquist Rate & Nyquist Interval

Over Sampling	Perfect Sampling	Under Sampling
$\omega_s > 2\omega_m$	$\omega_s = 2\omega_m$	$\omega_s < 2f_m$
$\cancel{2\pi} f_s > \cancel{2\pi} f_m$	$f_s = 2f_m$	$f_s < 2f_m$
$f_s > 2f_m$	$T = 1/f_s \Rightarrow T_s = \frac{1}{2f_m}$	
no overlapping	No Overlapping	Overlapping

Nyquist rate $= f_s = 2f_m$

Nyquist Interval $(T_s) = 1/f_s = 1/2f_m$

$m(t) = \cos 100\pi t + 2\sin 200\pi t$ more

T_s ? $\omega_1 = 100\pi t$

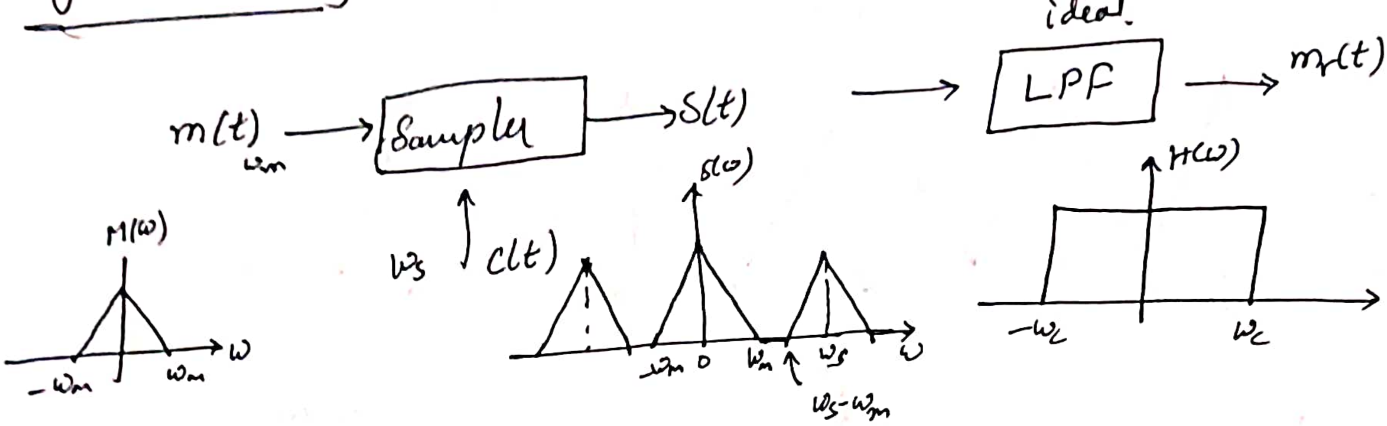
$\omega_2 = 200\pi t$ ✓

$\omega_2 = 2\pi f_2$

$f_2 = \frac{\cancel{200\pi t}\ 100}{\cancel{2\pi}}$

$f_2 = 100\ Hz$

Signal Recovery



$$\omega_s-\omega_m > \omega_m$$
$$\omega_s > 2\omega_m$$

$$H(\omega) = \frac{M_r(\omega)}{S(\omega)}$$
$$M_r(\omega) = S(\omega) . H(\omega)$$

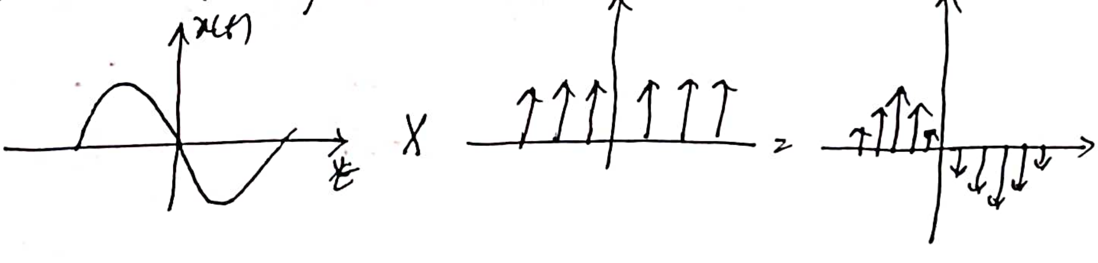
Types of Sampling

There are 3 types of sampling techniques

1) Impulse Sampling
2) Natural Sampling
3) Flat Top Sampling

Impulse Sampling: Impulse sampling can be performed by multiplying signal $x(t)$ with impulse train $\sum_{n=-\infty}^{\infty} \delta(t-nT)$ of period T. Here, the amplitude of impulse changes w.r.t. amplitude of input signal $x(t)$.

$$y(t) = x(t) \times \text{impulse train}$$



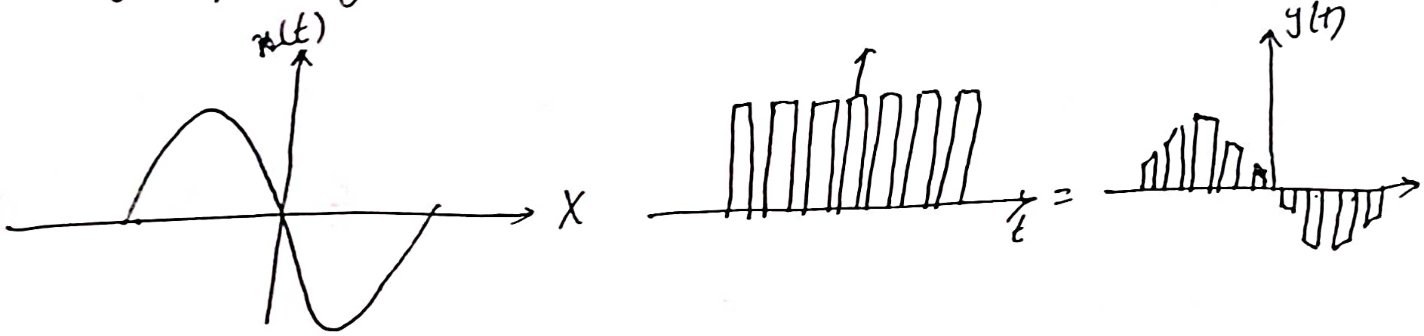
$$= x(t) \times \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

$$y(t) = y_s(t) = \sum_{n=-\infty}^{\infty} x(nt)\, \delta(t-nT)$$

This is called ideal Sampling or impulse sampling.

Natural Sampling

Natural Sampling is similar to impulse sampling, except the impulse train is replaced by pulse trains of period T. i.e you multiply input signal $x(t)$ to pulse train $\sum_{n=-\infty}^{\infty} P(t-nT)$



$$y(t) = x(t) \times \quad p(t)$$

$$= x(t) \times \sum_{n=-\infty}^{\infty} P(t-nT)$$

Exponential Fourier series of $p(t)$ can be

$$p(t) = \sum_{n=-\infty}^{\infty} F_n e^{j2\pi n f_s t}$$

where $$F_n = \frac{1}{T} \int_{-T/2}^{T/2} p(t)\, e^{-jn\omega_s t}\, dt$$

$$= \frac{1}{T\cancel{P}} P(n\omega_s)$$

$$p(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} P(n\omega_s)\, e^{jn\omega_s t}$$

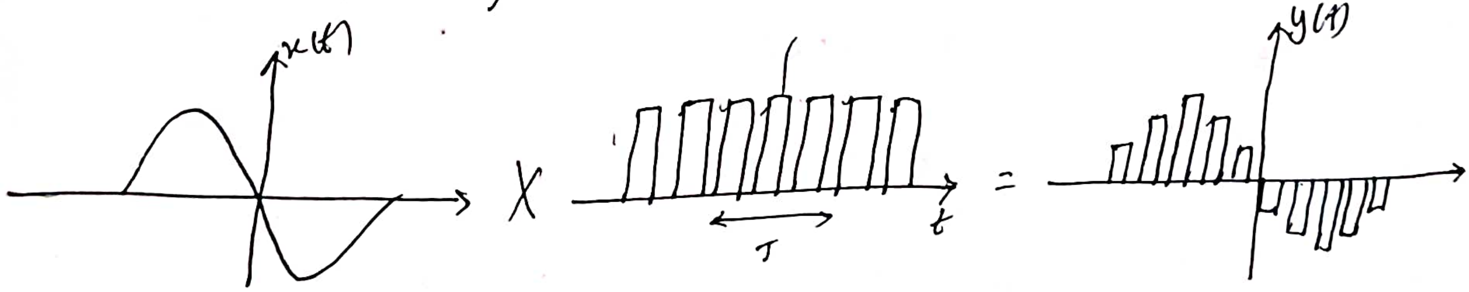
$$y(t) = x(t) \times \frac{1}{T} \sum_{n=-\infty}^{\infty} P(n\omega_s)\, e^{jn\omega_s t}$$

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} P(n\omega_s)\, x(t)\, e^{jn\omega_s t}$$

$$F.T\{y(t)\} = \frac{1}{T} \sum_{n=-\infty}^{\infty} P(n\omega_s)\, X(\omega - n\omega_s)$$

Flat Top Sampling

During transmission, noise is introduced at top of the transmission pulse which can be easily removed if the pulse is in the form of flat top. Here the top of the samples are flat. i.e they have constant amplitude. Hence it is called as Flat-Top Sampling or Practical Sampling. Flat-top sampling makes use of sample and hold circuit.



Convolution of rectangular pulse with ideally sampled signal

System:

$$\underset{x(t)}{\overset{i/p}{\longrightarrow}} \boxed{\text{System}} \underset{y(t)}{\overset{o/p}{\longrightarrow}}$$

$y(t) = x(t)$ – present
$y(t) = x(t-1)$ – past
$y(t) = x(t+1)$ – future.

1. Static and Dynamic Systems
2. Causal and Non-causal Systems
3. Time-Invariant and Time-Valiant Systems
4. Linear and Non-linear Systems
5. Invertible and Non-Invertable systems
6. Stable and Unstable Systems

Static and Dynamic Systems

* Static Systems → The o/p of the system depends only on present values of input.
* Dynamic Systems → The o/p of the system depends on past or future values of i/p at any instant of time.

$y(t) = 2.x(t)$

$t=0$ → indicates present state

so

System is static system

$y(t) = x(t+1) + x(t-1)$

$t=0$

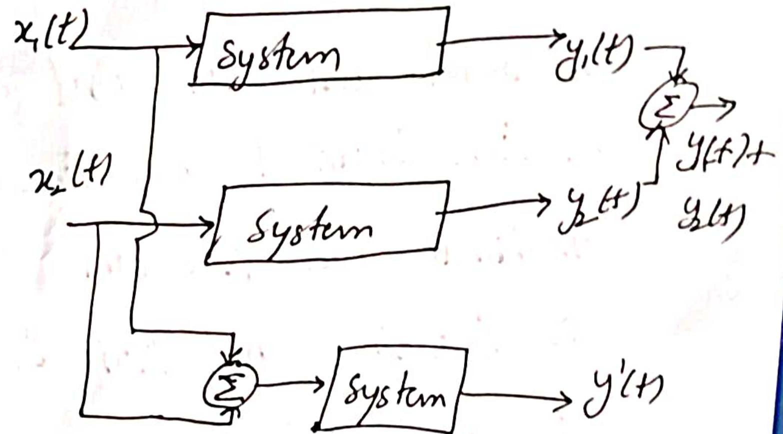
$y(t) = x(1) + x(-1)$ → o/p values are depend on past & future values so system is dynamic.

Linear and Non-Linear Systems

Linear System: The system which follows the principle of superposition is known as linear system.

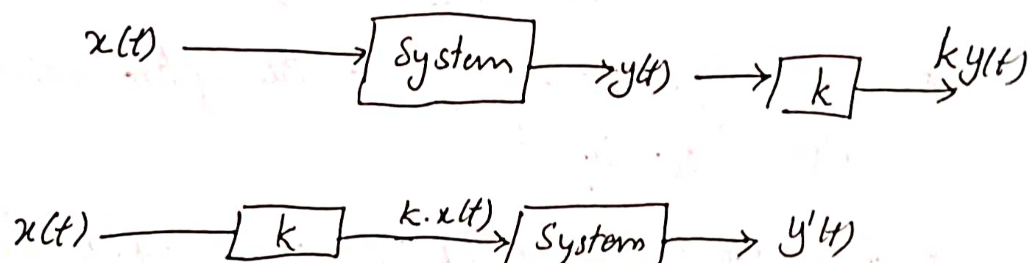
Superposition is a combination of two laws namely.

1. Law of additivity
2. law of homogenity



$y'(t) = y(t) \rightarrow$ additivity.

law of Homogenity



$\boxed{y'(t) = ky(t)}$

Eg: $y(t) = x(\sin t)$

Additivity

$y_1(t) = x_1(\sin t)$

$y_2(t) = x_2(\sin t)$

$y_1(t) + y_2(t) = x_1(\sin t) + x_2(\sin t)$

$x_1(t) + x_2(t) \longrightarrow \text{system} \rightarrow y(t)$

$x_1(\sin t) + x_2(\sin t) \rightarrow$

homogenity $k.y(t) = kx(\sin t)$

$kx(\sin t) =$

static systems are systems without memory

dynamic systems are systems with memory

---

1. $y(t) = x(t^2)$

$y_1(t) = x_1(t^2)$
$y_2(t) = x_2(t^2)$ } $y_1(t) + y_2(t) = x_1(t^2) + x_2(t^2)$ — 1

$x_1(t) + x_2(t) \rightarrow$ system — $x_1(t^2) + x_2(t^2)$ — 2

$k\, x(t^2)$

System linearity is independent of time scaling

Time scaling

$y(t) = x^2(t)$

$x(t) \longrightarrow$ system $y(t) = x^2(t)$

$y_1(t) \rightarrow x_1^2(t)$
$y_2(t) \rightarrow x_2^2(t)$

$y_1(t) + y_2(t) \longrightarrow x_1^2(t) + x_2^2(t)$

$(x_1(t) + x_2(t))^2$

$\boxed{y(t) = x^3(t^3)}$?

i) $y(t) = \cos[x(t)]$

ii) $y(t) = x(t+1)e^{-t}$ ✓

iii) $y(t) = 3x(t+3)$

iv) $y(t) = t \cdot x(t)$

Causal system: o/p of system is independent of future values of i/p

o/p is dependent on past and previous i/p values

all practical systems are causal.

Non-causal system is dependent on future inputs.

Stable and unstable systems

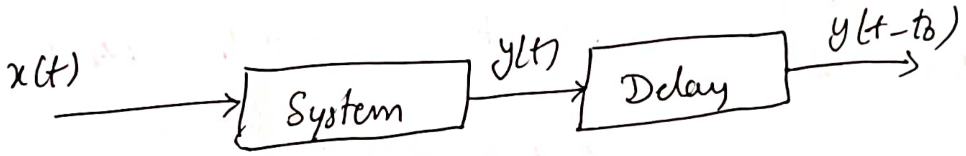
BIBO Criteria → Bounded Input and Bounded Output

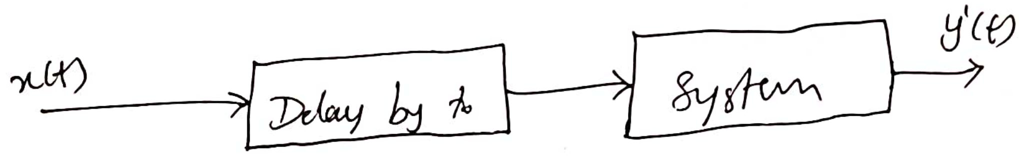
Stable systems are having the o/p should be bounded for bounded i/p. at each and every instant of time.

Eg: dc value. sint. cost, u(t)

amplitude is finite in every instant of time from $-\infty$ to ∞

Time Variant & Time Invariant Systems





$y'(t) = y(t-t_0)$ TIV systems

$y'(t) \neq y(t-t_0)$ TV systems

1) $y(t) = x(2t)$

$x(t) \longrightarrow$ Syst. $\longrightarrow x(2t) = y(t)$

S-1 $y(t) \longrightarrow y(t-t_0) = x[2(t-t_0)] = x(2t-2t_0)$

$x(t) \longrightarrow x(t-t_0) \rightarrow$ Syst $\longrightarrow x(2t-t_0)$

2) $y(t) = 2 + x(t)$

$x(t) \longrightarrow sys \longrightarrow 2 + x(t) = y(t)$

$y(t) \xrightarrow{t_0} 2 + x(t - t_0)$

$x(t - t_0) \longrightarrow sys \longrightarrow 2 + x(t - t_0)$

TIV system

3) $y(t) = x(\cos t)$

$x(t) \longrightarrow \boxed{sys} \longrightarrow y(t) = x(\cos t)$

$y(t) \xrightarrow{t_0} y(t - t_0) = x[\cos(t - t_0)]$

$x(t) \xrightarrow{t_0} x(t - t_0) \longrightarrow \boxed{Sys} \longrightarrow x[\cos t - t_0]$

TV system

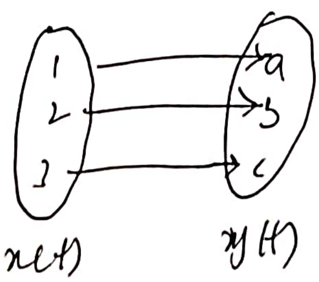
4) $y(t) = x(\tan t)$

TV System

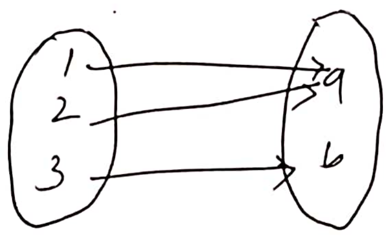
Invertable and Non Invertable Systems:

For an invertable system, there should be one to one mapping b/w i/p and o/p at each other and every instant of time.

one to one mapping

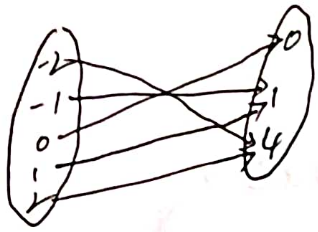


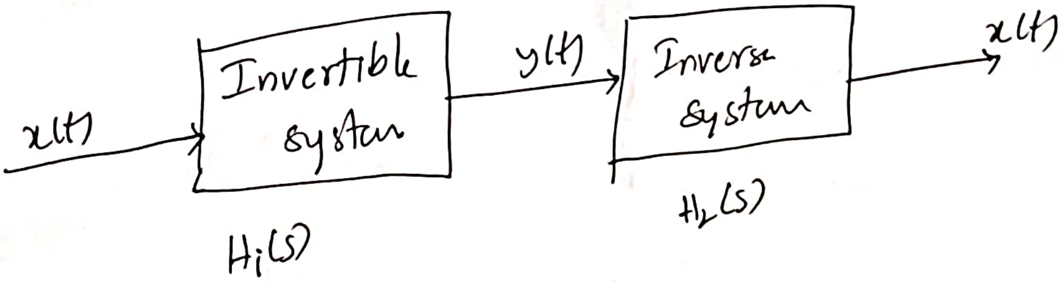
one to many



Ex: $y(t) = x^2(t)$

Ex 2: $y(t) = x(t) + 2$





$$H(s) = 1 = H_1(s) \times H_2(s)$$

$y(t) = |x(t)|$

$y(t) = \sin t \cdot x(t)$

$y(t) = x(\sin t)$

$y(-\pi) = x(\sin(-\pi))$

$= x(0)$

Here $y(-\pi) = x(0)$

it depends on future values

so non-causal

Filter Characteristics of Linear Systems

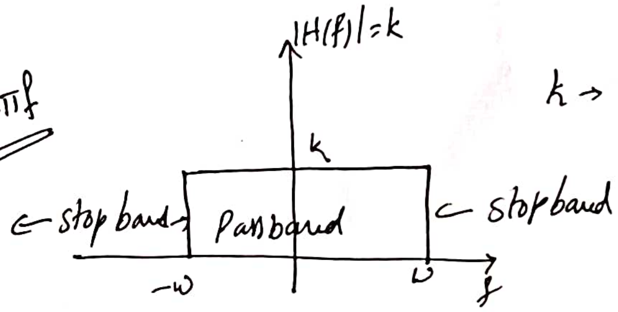
An Ideal LPF transmits all of the signals below certain frequency 'w' Hz. without any distortion. The range $-w$ to $+w$ frequency called passband. w is called as cutoff frequency.

T.F of ideal LPF

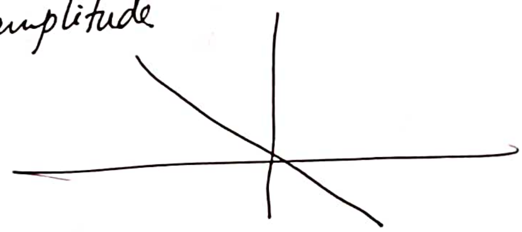
$$H(\omega) = k e^{-j\omega t_0} \qquad H(f) = k e^{-j2\pi f t_0} \qquad -\omega_c \le 2f < \omega_c$$

$$= 0 \qquad |f| > \omega_c$$

$\omega = 2\pi f$



$k \rightarrow$ amplitude



$$h(t) = \int_{-w}^{w} e^{-j2\pi f t_0} \cdot e^{j2\pi f t} \cdot df \quad \text{(or)} \quad h(t) = \int_{-\omega_c}^{\omega_c} e^{-j\omega t_0} \cdot e^{j\omega t} \cdot d\omega$$

$$= \int_{-\omega_c}^{\omega_c} e^{j\omega(t-t_0)} \cdot d\omega$$

$$= \frac{1}{j(t-t_0)} \left[e^{j\omega(t-t_0)} \right]_{-\omega_c}^{\omega_c} = \frac{2}{t-t_0} \left[\frac{e^{j\omega_c(t-t_0)} - e^{-j\omega_c(t-t_0)}}{2j} \right]$$

$$= 2 \sin \frac{\omega_c(t-t_0)}{t-t_0} = 2\omega_c \left(\frac{\sin \omega_c (t-t_0)}{\omega_c(t-t_0)} \right)$$

$$\boxed{= 2\omega_c \, \mathrm{sinc}(\omega_c(t-t_0))}$$

Magnitude distortion

If the system provides unequal amount of amplification to different frequency components available in input signal then system having magnitude distortion.

$$x(t) \longrightarrow \boxed{x(t)} \longrightarrow y(t) = A_1 \sin \omega_1 t + A_2 \sin \omega_2 t$$

$\sin \omega_1 t$

$\sin \omega_2 t$

$\omega_1 \neq \omega_2$

$A_1 \neq A_2$

Magnitude distortion occuring

Phase distortion

If the system provides unequal amount of time delays to different frequency components available in input signal then system is having phase distortion.

$$x(t) \longrightarrow \boxed{LTI} \longrightarrow y(t)$$

$\sin \omega_1 t$

$\sin \omega_2 t$

$= \sin \omega_1 (t_1 - t_2) + \sin \omega_2 (t - t_2)$

$t_1 \neq t_2$ then

Distortion less LTI System

$$x(t) \longrightarrow \boxed{\begin{array}{c}\text{LTI}\\ \text{System}\end{array}} \longrightarrow y(t)$$

let

$$\sin \omega_1 t + \sin \omega_2 t \qquad \omega_1 \neq \omega_2$$

$$y(t) = k_1 \sin(\omega_1(t - t_1)) + k_2 \sin(\omega_2(t - t_2))$$

magnitude distortion , phase distortion

$k_1 = k_2 = k \rightarrow$ to avoid distortion amplification must be same

$t_1 = t_2 = t_0 \rightarrow$ to avoid distortion in phase.

$$y(t) = k \sin \omega_1(t - t_0) + k \sin \omega_2(t - t_0)$$

$$= k \cdot x(t - t_0)$$

L.T on both sides

$$Y(s) = k \cdot X(s) \cdot e^{-st_0}$$

$$\frac{Y(s)}{X(s)} = H(s) = ke^{-st_0} \xrightarrow{s = j\omega} H(j\omega) = ke^{-j\omega t_0}$$

$\boxed{H(\omega) = ke^{-j\omega t_0}} \rightarrow$ Transfer function for distortionless LTI system

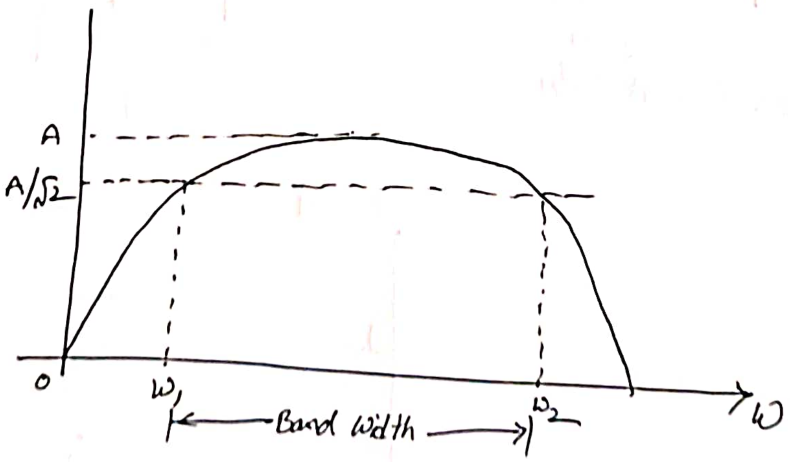
Signal Bandwidth:

The spectral components of a signal extend from $-\infty$ to ∞. Any practical signal has finite amount of energy. The spectral components approach zero as ω tends to ∞. So we neglect the spectral components which have negligible energy and select only a band of frequency components which have most of the signal energy.

The band of frequencies that contain most of the signal energy is known as the bandwidth of the signal.

System bandwidth:

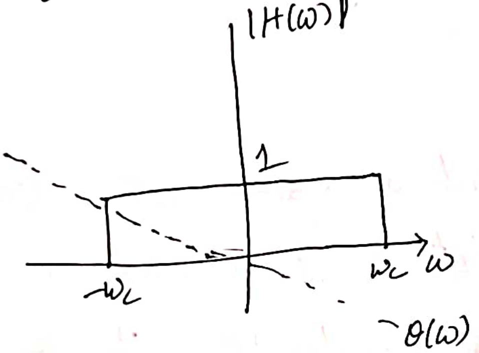
For distortionless transmission we need a system with infinite bandwidth. Due to physical limitations, it is impossible to construct a system with infinite bandwidth. The bandwidth of a system is defined as the range of frequencies over which the magnitude $|H(\omega)|$ remains within $1/\sqrt{2}$ times of its value at midband.



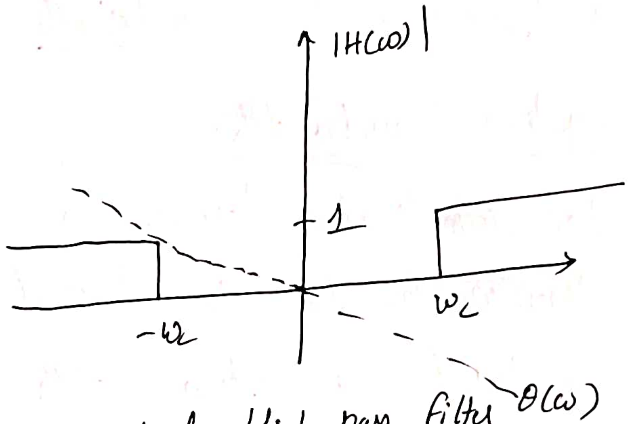
Ideal Filter Characteristics

An ideal filter has very sharp cutoff characteristics and it passes signals of certain specified band of frequencies exactly and totally rejects signals of frequencies outside this band.

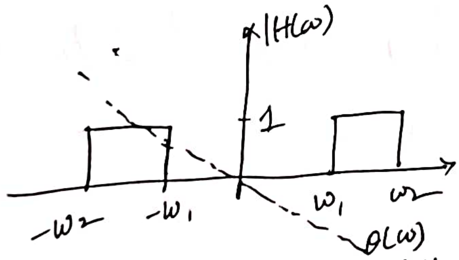
Filters are usually classified according to their frequency response characteristics as Low Pass filter (LPF), High-Pass filter (HPF), Band Pass filters (BPF) and Band Elimination (or) Band stop (or) Band Reject filter.



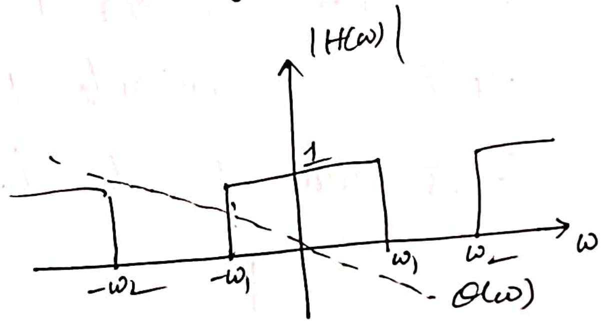
(a) ideal Low Pass filter



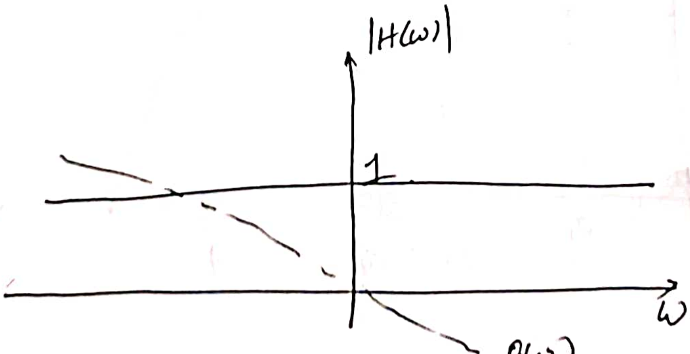
ideal High pass filter



ideal Band pass filter



ideal Band Reject filter



ideal all pass filter

Ideal LPF

An ideal Low pass filter transmits, without any distortion, all of the signals of frequencies below a certain frequency ω_c, radians per second.

$$|H(\omega)| = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

Ideal HPF

An ideal high pass filter transmits, without any distortion, all of the signals of frequencies above a certain frequency ω_c, rad/sec

$$|H(\omega)| = \begin{cases} 0 & |\omega| < \omega_c \\ 1 & |\omega| > \omega_c \end{cases}$$

Ideal BPF

An ideal band pass filter transmits, without any distortions all of the signals of frequencies with in a certain frequency band $(\omega_2 - \omega_1)$ rad/sec

$$|H(\omega)| = \begin{cases} 1 & |\omega_1| < \omega < |\omega_2| \\ 0 & \omega < |\omega_1| \text{ and } \omega > |\omega_2| \end{cases}$$

Ideal BRF

An ideal Band Reject filter rejects totally all of the signals of frequencies within a certain frequency band $(\omega_2 - \omega_1)$ rad/sec and transmits without any distortion all signals of frequencies outside this band $(\omega_2 - \omega_1)$ is the rejection band.

$$|H(\omega)| = \begin{cases} 0 & |\omega_1| < \omega < |\omega_2| \\ 1 & \omega < |\omega_1| \text{ and } \omega > |\omega_2| \end{cases}$$

Paley - Wiener Criterion for physical realization

A system is said to be causal if it does not produce an output before the input is applied.

$$h(t) = 0 \text{ for } t < 0.$$

Physical realizability implies that it is physically possible to construct that system in real time. In frequency domain the physical realizability criterion implies that a necessary and sufficient condition for a magnitude function $H(\omega)$ to be physically realizable is:

$$\int_{-\infty}^{\infty} \frac{\ln |H(\omega)|}{1+\omega^2} d\omega < \infty$$

The magnitude function $|H(\omega)|$ must, however be square-integrable before the Paley-Wiener criterion is valid. ie:

$$\int_{-\infty}^{\infty} |H(\omega)|^2 d\omega < \infty.$$

The Paley-Wiener criterion:

1. The magnitude function $|H(\omega)|$ may be zero at some discrete frequencies, but it cannot be zero over a finite band of frequencies since this will cause the integral in the equation of Paley-wiener criterion to become infinity. That means ideal filters are not physically realizable.

2. The magnitude function $|H(\omega)|$ cannot fall off to zero faster than a function of exponential order. It implies a realizable magnitude characteristic cannot have too great a total attenuation.

Relationship b/w Bandwidth and Rise Time

The transfer function of ideal low pass filter is given by

$$H(\omega) = |H(\omega)| e^{-j\omega t_d}$$

$$|H(\omega)| = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$ ω_c is cutoff frequency

$\therefore$

$$H(\omega) = e^{-j\omega t_d} \qquad -\omega_c \le \omega \le \omega_c \quad \text{i.e } |\omega| \le \omega_c$$

$$= 0 \qquad \omega > |\omega_c|$$

The impulse response $h(t)$ of the LPF is obtained by taking the inverse fourier transform of the T.F $H(\omega)$

$$h(t) = \mathcal{F}^{-1}[H(\omega)] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega t_d} \cdot e^{j\omega t} \cdot d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(t-t_d)} \cdot d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega_c(t-t_d)} - e^{-j\omega_c(t-t_d)}}{j(t-t_d)}\right]$$

$$= \frac{1}{\pi(t-t_d)}\left[\sin\omega_c(t-t_d)\right] \quad \because \sin\omega_c(t-t_d) = \frac{e^{j\omega_c(t-t_d)} - e^{-j\omega_c(t-t_d)}}{2j}$$

$$= \frac{\omega_c}{\pi}\left[\frac{\sin\omega_c(t-t_d)}{\omega_c(t-t_d)}\right] = \frac{\omega_c}{\pi}\,\text{sinc}\ \omega_c(t-t_d)$$

$$\boxed{h(t) = \frac{\omega_c}{\pi}\,\text{sinc}\ \omega_c(t-t_d)}$$

The step response $y(t) = h(t) * u(t) = \int_{-\infty}^{t} h(\tau)\,d\tau$.

$$y(t) = \int_{-\infty}^{t} \frac{w_c}{\pi} \frac{\sin w_c(\tau - t_d)}{w_c(\tau - t_d)}\,d\tau.$$

$$x = w_c(\tau - t_d)$$

$$dx = w_c\,d\tau \quad \text{or} \quad d\tau = \frac{dx}{w_c}$$

$$y(t) = \int_{-\infty}^{w_c(t-t_d)} \frac{\cancel{w_c}}{\pi} \frac{\sin x}{x} \cdot \frac{dx}{\cancel{w_c}} = \frac{1}{\pi} \int_{-\infty}^{w_c(t-t_d)} \frac{\sin x}{x} \cdot dx$$

$$= \frac{1}{\pi} \left[Si(x) \right]_{-\infty}^{w_c(t-t_d)}$$ where Si is the sine integral function

1. $Si(x)$ is an odd function, that is $Si(-x) = -Si(x)$
2. $Si(0) = 0$
3. $Si(\infty) = \pi/2$ and $Si(-\infty) = (-\pi/2)$

$$y(t) = \frac{1}{\pi} \{ Si(w_c(t-t_d)) - Si(-\infty) \}$$

$$= \frac{1}{\pi} \{ Si[w_c(t-t_d)] + \frac{\pi}{2} \}$$

$$= \frac{1}{2} + \frac{1}{\pi} \sin[w_c(t-t_d)]$$

if $w_c \to \infty$, then the response is

$$y(t) = \frac{1}{2} + \frac{1}{\pi} Si(\infty) = \frac{1}{2} + \frac{1}{\pi}\left(\frac{\pi}{2}\right) = 1$$

$w_c \to -\infty$ then the response is

$$y(t) = \frac{1}{2} + \frac{1}{\pi} Si(-\infty) = \frac{1}{2} + \frac{1}{\pi}\left(-\frac{\pi}{2}\right) = 0$$

The rise time t_r is defined as the time required for the responses to reach from 0% to 100% of the final value.

at $t = t_d$ the line $y(t) = 0$ and $y(t) = 1$

$$\frac{d}{dt} y(t)\Big|_{t=t_d} = \frac{1}{t_r} = \frac{\omega_c}{\pi} \frac{\sin \omega_c (t-t_d)}{\omega_c (t-t_d)}\Big|_{t=t_d}$$

$$\boxed{t_r = \frac{\pi}{\omega_c}}$$

Bandwidth $\times$ Rise time = Constant

Ex: Let the system function of an LTI System be $1/(j\omega+2)$. what is the output of the system for an input $(0.8)^t u(t)$.

Sol: Given transfer function $H(\omega) = \dfrac{1}{j\omega+2}$

$$x(t) = (0.8)^t . u(t)$$

$$h(t) = \mathcal{F}^{-1}\left[\frac{1}{j\omega+2}\right] = e^{-2t} u(t)$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau) . x(t-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2t} u(t) (0.8)^{t-\tau} u(t-\tau) \, d\tau$$

$$= \int_0^{\infty} e^{-2t} . (0.8)^t (0.8)^{-\tau} . d\tau = 0.8^t \int_0^{\infty} (0.8e^2)^{-\tau} . d\tau$$

$$\because \int a^x dx = \frac{a^x}{\log a} \quad \text{So. } y(t) = (0.8)^t \left[\frac{(0.8e^2)^{-\tau}}{\log (0.8e^2)^{-1}}\right]_0^{\infty}$$

$$= \frac{0.8^t}{-\log\left[(0.8e^2)^{-1}\right]}\left[(0.8e^2)^0 - 1\right]$$

$$= \frac{(0.8)^t}{(\log 0.8 + 2)}$$

A convolution is an integral that expresses the amount of overlap of one function when it is shifted over another function.

(or)

Convolution is used to find the common area b/w two signals.

1. Shifting ($t \pm t_0$)
2. Scaling (reversal)
3. Differentiation
4. Integration
5. Convolution

$$f(t) = f_1(t) * f_2(t)$$

$$= \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau$$

let $x(t)$ is input signal and $h(t)$ is system response then o/p signal is $y(t)$.

$$y(t) = x(t) * h(t)$$

if $x(t)$ and $h(t)$ are non causal

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

if $x(t)$ is non causal and $h(t)$ is causal

$$y(t) = \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau.$$

If $x(t)$ is causal and $h(t)$ is noncausal

$$y(t) = \int_{0}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

if $x(t)$ and $h(t)$ are causal

$$y(t) = \int_{0}^{t} x(\tau)\, h(t-\tau)\, d\tau.$$

Convolution Theorem

The time convolution states that the convolution in time-domain is equivalent to multiplication of their spectra in frequency domain. Mathematically if

$$x_1(t) \longleftrightarrow X_1(\omega)$$

$$x_2(t) \longleftrightarrow X_2(\omega)$$

$$x_1(t) * x_2(t) \longleftrightarrow X_1(\omega) \cdot X_2(\omega)$$

$$\mathcal{F}\{x_1(t) * x_2(t)\} = \int_{-\infty}^{\infty} [x_1(t) * x_2(t)] \cdot e^{-j\omega t} \cdot dt \quad \text{--- (1)}$$

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \quad \text{--- (2)}$$

Substituting eq (2) in eqn (1)

$$\mathcal{F}\{x_1(t) * x_2(t)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau\, e^{-j\omega t} \cdot dt$$

interchanging order of integration

$$= \int_{-\infty}^{\infty} x_1(\tau) \cdot \int_{-\infty}^{\infty} x_2(t-\tau) \cdot e^{-j\omega t} \cdot dt \cdot d\tau$$

$t = k+\tau$ and $dt = dk$

$$= \int_{-\infty}^{\infty} x(\tau) \cdot \underbrace{\int_{-\infty}^{\infty} x_2(k) \cdot e^{-j\omega k} \cdot dk}_{X_2(\omega)} \cdot e^{-j\omega \tau} \cdot d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) \cdot e^{-j\omega\tau} \cdot d\tau \cdot X_2(\omega) = X_1(\omega) \cdot X_2(\omega)$$

Hence Proved

Properties of Convolution

1. **Commutative Property:**

$$x_1(t) * x_2(t) = y(t) = x_2(t) * x_1(t)$$

$x_1(t)$: fix $x_1(\tau)$; $x_2(t)$: move $x_2(t-\tau)$; $x_2(t)$: $\underbrace{x_2(\tau)}_{\text{fix}}$; $x_1(t)$: move $\underbrace{x_1(t-\tau)}_{\text{Mov}}$

$\boxed{h(t)}$

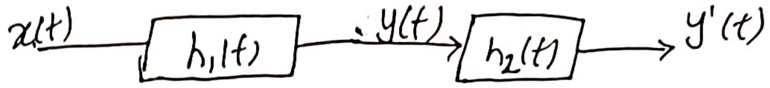
Impulse response is used give the response of LTI system

$$y(t) = x(t) * h(t) = h(t) * x(t)$$

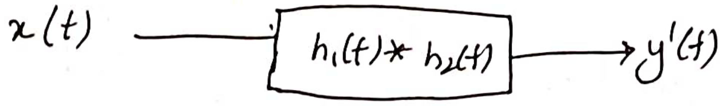
$x(t)$: Input; $h(t)$: Impulse response; $h(t)$: i/p; $x(t)$: Impulse response

2. **Associative Property:** $x_1(t)$, $x_2(t)$, $x_3(t)$

$$(x_1(t) * x_2(t)) * x_3(t) = y(t) = x_1(t) * (x_2(t) * x_3(t))$$



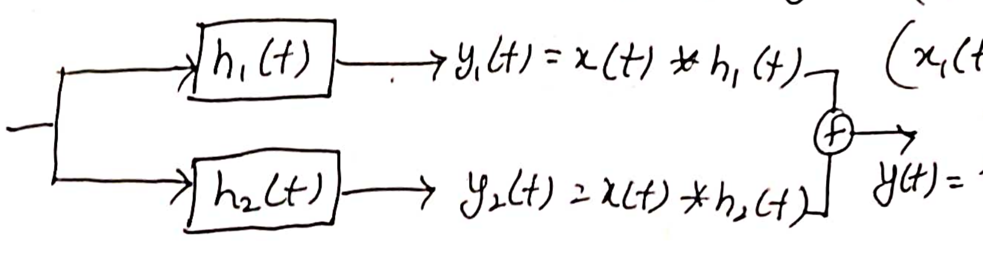
$$y'(t) = (x(t) * h_1(t)) * h_2(t)$$



$$= x(t) * (h_1(t) * h_2(t))$$

3. **Distributive Property**

$$x_1(t) * (x_2(t) + x_3(t)) = y(t) = (x_1(t) * x_2(t)) + (x_1(t) * x_3(t))$$



$$x(t) \longrightarrow \boxed{h_1(t)+h_2(t)} \longrightarrow y(t) = x(t) * (h_1(t)+h_2(t))$$

4 Property of Delta function

$$x(t) * \delta(t-t_1) = x(t-t_1)$$

$$\downarrow t_1 = 0$$

$$x(t) * \delta(t) = x(t)$$

$$x(t) * A\delta(t-t_1) = Ax(t-t_1)$$

$$\downarrow t_1 = 0$$

$$x(t) * A\delta(t) = Ax(t)$$

ex: 1 $\quad x(t) = r(t)$

$$r(t) * \delta(t-2) = r(t-2)$$

5 Derivative property

$$y(t) = x(t) * h(t)$$

$$\frac{d}{dt} y(t) = \frac{d}{dt}[x(t) * h(t)]$$

$$= x(t) * \frac{d\,h(t)}{dt} \quad \text{— I}$$

$$= \frac{d\,x(t)}{dt} * h(t) \quad \text{— II}$$

Eg: Method-I

$$y(t) = r(t) * u(t)$$

$$\frac{d}{dt} y(t) = r(t) * \frac{d\,u(t)}{dt}$$

$$= r(t) * \delta(t)$$

$$= r(t)$$

Method-II

$$\frac{d\,y(t)}{dt} = \frac{d\,r(t)}{dt} * u(t)$$

$$= u(t) * u(t)$$

$$= r(t)$$

6. Integration

$$I[D(y(t))] = y(t)$$

$$y(t) = x(t) * u(t) = \int_{-\infty}^{t} x(\tau) \cdot d\tau$$

Proof

$$y(t) = x(t) * u(t)$$

$$\frac{d}{dt} y(t) = x(t) * \frac{du(t)}{dt}$$

$$\frac{d}{dt} y(t) = x(t) * \delta(t)$$

$$\int_{-\infty}^{t} \frac{d}{dt} y(t) = \int_{-\infty}^{t} x(\tau) * \delta(\tau)$$

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau.$$

7 Time delay

$$x_1(t) * x_2(t) = y(t)$$

$$\Big\downarrow \text{delay } t_1 \qquad \Big\downarrow \text{delay } t_2$$

$$x_1(t-t_1) * x_2(t-t_2) = y[t-(t_1+t_2)]$$

Linear Convolution

$x(n) = [1, 2, 3, 4]$ no. of Samples (L) = 4

$h(n) = (-3, 2, 1)$ no. of Samples (M) = 3

Length of linear Convolution Samples N = M+L-1
= 4+3-1 = 6

	1	2	3	4
-3	-3	-6	-9	-12
2	2	4	6	8
1	1	2	3	4

$y[n] = \{-3, -4, -4, -4, 11, 4\}$

Circular

$$\begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ -4 \\ -4 \end{bmatrix}$$

Correlation:

Correlation of two signals is a measure of similarity b/w those signals.

$$R(\tau) = \int_{-\infty}^{\infty} x_1(t)x_2(t-\tau)dt \qquad x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau.$$

Correlation is of two types

1) Auto Correlation $\quad R_{11}(\tau)$

2) Cross Correlation $\quad R_{12}(\tau), R_{21}(\tau)$

Auto correlation function gives measure of math (or) similarity (or) relatedness (or) coherence b/w a signal & its time shifted version

$$R_{11}(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt = \int_{-\infty}^{\infty} x(t-\tau)x^*(t)dt$$

Where τ = Searching (or) Scanning (or) delay parameter.

If signal is real Then

$$R_{11}(\tau) = \int_{-\infty}^{\infty} x(\tau).x(t-\tau)dt \rightarrow \text{Aperiodic ACF}$$

Properties of autocorrelation

1) Autocorrelation function of power signal exhibits conjugate Symmetry.

$$R(\tau) = R^*(-\tau)$$

2) Auto correlation function at origin is equal to power of that signal

$$R(0) = P.$$

3) ACF is maximum at $\tau = 0$ & increases with decrease of τ & vice versa

i.e $|R(\tau)| \leq R(0)$

4) ACF and PSD are Fourier transform Pair

$$R(\tau) \xleftrightarrow{F.T} S(\omega)$$

$$\boxed{R(\tau) = \lim_{T \to \infty} \int_{-T/2}^{T/2} x(t) x^*(t+\tau) \, dt.}$$ → Periodic signals ACF

Properties of cross correlation function

1) CCF exhibit Conjugate Symmetry

$$R_{12}(\tau) = R_{21}^*(-\tau)$$

2) if $R_{12}(0) = 0$ then the signals are said to be orthogonal.

3) $R_{12}(\tau) \xleftrightarrow{FT} X_1(\omega) X_2^*(\omega)$

$$R_{21}(\tau) = \int_{-\infty}^{\infty} x_1^*(t) \, x_2(t-\tau) . dt = \int_{-\infty}^{\infty} x_1^*(\tau) x_2(t+\tau) dt$$

$$R_{21}^*(-\tau) = \int_{-\infty}^{\infty} x_1(\tau) \, x_2^*(t+\tau) dt$$

$$= R_{12}(\tau).$$

1) The impulse response of continuous time system is given as

$$h(t) = \frac{1}{RC} e^{-t/RC} \cdot u(t)$$

Determine the frequency response & plot the magnitude phase plots.

Sol:

$$H(\omega) = \int_{-\infty}^{\infty} h(t) \cdot e^{-j\omega t} \cdot dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{RC} \cdot e^{-t/RC} \cdot u(t) \cdot e^{-j\omega t} \cdot dt$$

$u(t) = 1$ for $t > 0$
0 otherwise

$$= \frac{1}{RC} \int_{0}^{\infty} e^{-t/RC} \cdot e^{-j\omega t} \cdot dt$$

$$= \frac{1}{RC} \int_{0}^{\infty} e^{-t(j\omega + \frac{1}{RC})} \cdot dt$$

$$= \frac{1}{RC} \left[\frac{-1}{j\omega + \frac{1}{RC}} \right] \left[e^{-t(j\omega + 1/RC)} \right]_{0}^{\infty}$$

$$H(\omega) = \frac{1/RC}{j\omega + 1/RC} = \frac{1}{1 + j\omega RC}$$

Magnitude response & Phase Response

$$H(\omega) = \frac{1}{1 + j\omega RC} \times \frac{1 - j\omega RC}{1 - j\omega RC} = \frac{1 - j\omega RC}{1 + (\omega RC)^2}$$

$$= \frac{1}{1+(\omega RC)^2} + j\,\frac{-\omega RC}{1+(\omega RC)^2}$$

$$|H(\omega)| = \left\{ \frac{1}{[1+(\omega RC)^2]^2} + \frac{(\omega RC)^2}{[1+(\omega RC)^2]^2} \right]^{1/2}$$

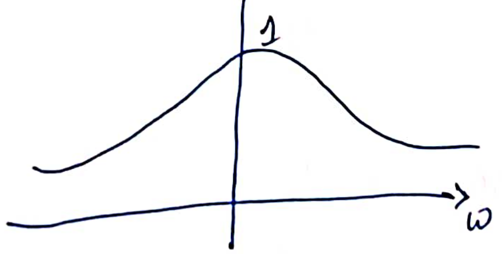
$$= \frac{1}{\sqrt{1+(\omega RC)^2}}$$

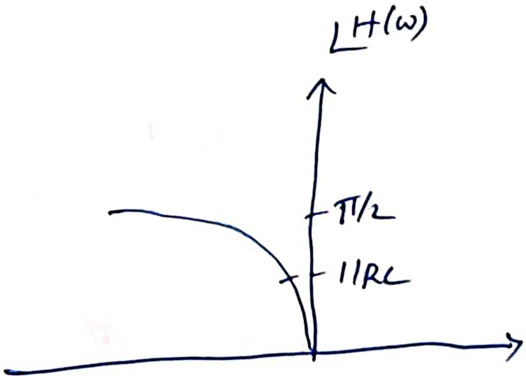
$$\angle H(\omega) = \tan^{-1}\left\{ \frac{-\omega RC/(1+(\omega RC)^2)}{1/[1+(\omega RC)^2]} \right.$$

$$= \underline{-\tan^{-1}(\omega RC)}$$

If $RC = 1$

$$|H(\omega)| = \frac{1}{\sqrt{1+\omega^2}} \qquad \angle H(\omega) = -\tan^{-1}(\omega)$$

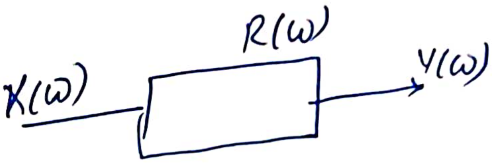




2) For the system find the impulse response of the system

$$x(t) = e^{-at} \quad t > 0$$
$$= 0 \quad \text{elsewhere}$$

$$Y(\omega) = \frac{1}{\alpha + j\omega}$$



$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

$$X(\omega) = \frac{1}{a + j\omega}, \quad Y(\omega) = \frac{1}{\alpha + j\omega}$$

$$H(\omega) = \frac{1/\alpha + j\omega}{1/a + j\omega} = \frac{a + j\omega}{\alpha + j\omega}$$

$$F^{-1}\left[\frac{a + j\omega}{\alpha + j\omega}\right] = \frac{a + \alpha - \alpha + j\omega}{\alpha + j\omega} = \frac{a - \alpha}{\alpha + j\omega} + \frac{\alpha + j\omega}{\alpha + j\omega}$$

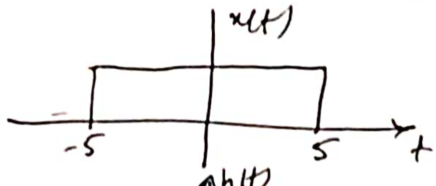
$$= \frac{a - \alpha}{\alpha + j\omega} + 1$$

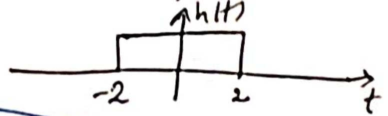
$$\boxed{h(t) = (a - \alpha) \cdot e^{-\alpha t} u(t) + \delta(t)}$$

Q:

$x(t) = A \quad -5 \le t \le 5$

$h(t) = A \quad -2 \le t \le 2$



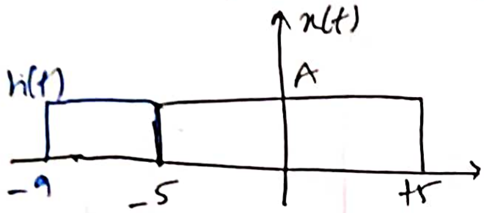


Perform Convolution using graphical method

no. of limits = 14 = -7 to 7

$t = -7$

$x(\tau)$ is fixed $\quad h(\tau)$ = moving

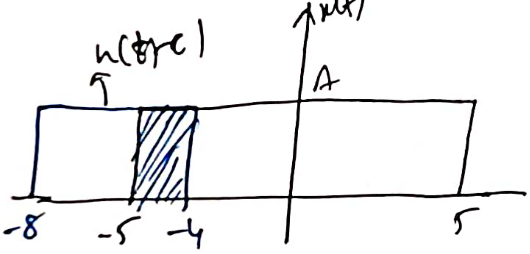


$h(t-\tau) = h(-7-2) = h(-9)$

$h(-7-(-2)) = h(-7+2) = h(-5)$

$$\int_{-9}^{-5} 0.0 \cdot dt = \underline{\underline{0}}$$

$t = -6$



$h(t-\tau) = h(-6-(-2)) = h(-4)$

$h(t-\tau) = h(-6-2) = h(-8)$

$$\int_{-5}^{-4} A \cdot A \cdot dt = A^2 \Big[t \Big]_{-5}^{-4}$$

$$= \underline{\underline{A^2}}$$

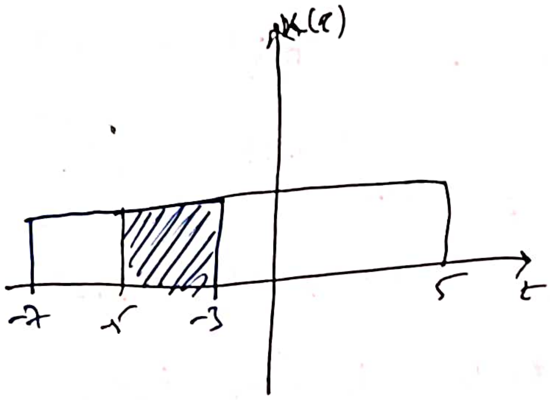
$\underline{\underline{t = -5}}$

$\tau = +2$

$h(t-\tau) = h(-5-2) = h(-7)$

$\tau = -2$

$h(t-\tau) = h(-5+2) = h(-3)$



$$\int_{-5}^{-3} A \cdot A \, dt = A^2[-3-(-5)]$$

$$= \underline{\underline{2A^2}}$$

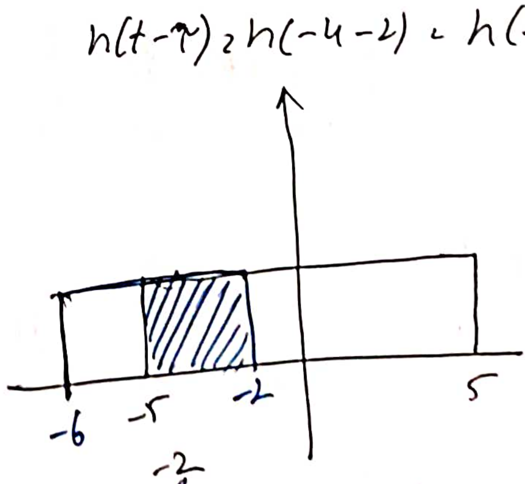
$\underline{t = -4}$

at $\tau = -2$

$h(t-\tau) = h(-4-(-2)) = h(-2)$

at $\tau = +2$

$h(t-\tau) = h(-4-2) = h(-6)$



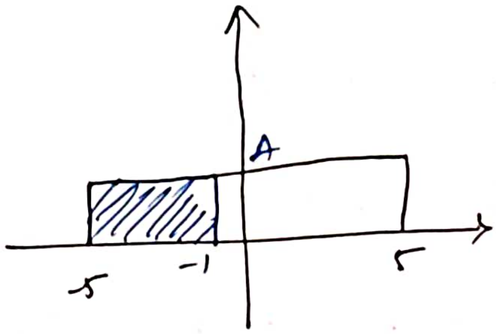
$$\int_{-5}^{-2} A \cdot A \, dt = A^2[-2-(-5)]$$

$$= \underline{3A^2}$$

at $t=-3$

$\tau=-2$ then $h(-3-(-2)) = h(-1)$

$\tau = 2$ $\quad h(-3-2) = h(-5)$



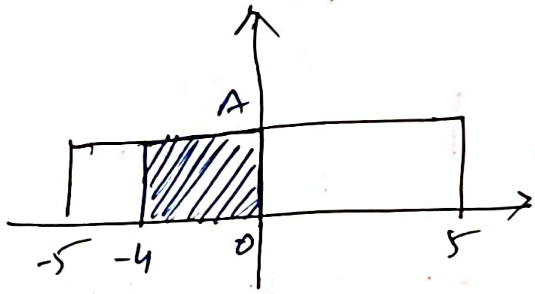
$$\int_{-5}^{-1} A \cdot A \cdot dt = A^2[t]_{-5}^{-1} = A^2[-1-(-5)] = 4A^2$$

at $t=-2$

$\tau=-2$ $\quad h(t-\tau) = h(-2-(-2)) = h(0)$

$\tau = 2$ $\quad h(t-\tau) = h(-2-2) = h(4)$

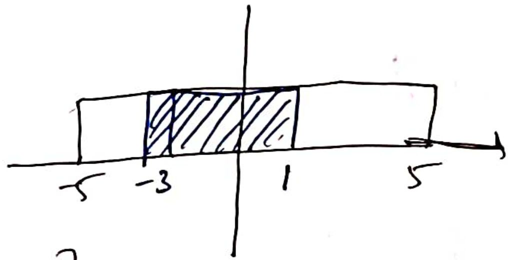
$$\int_{-4}^{0} A \cdot A \cdot dt = A^2[t]_{-4}^{0} = 4A^2$$



at $t=-1$

$\tau=-2$ $\quad h(t-\tau) = h(-1+2) = h(+1)$

$\tau=+2$ $\quad h(-1-2) = h(-3)$

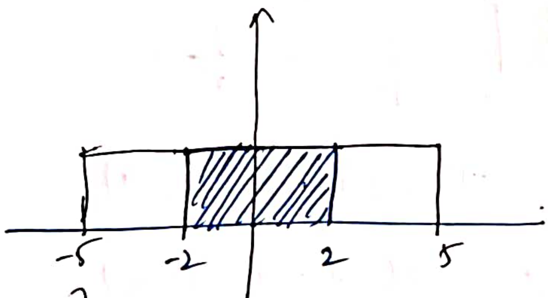


$$\int_{-3}^{1} A \cdot A \cdot dt = A^2[t]_{-3}^{1} = A^2[1-(-3)]$$
$$= 4A^2$$

$t=0$

$\tau=-2$ $\quad h(0-(-2)) = h(2)$

$\tau = 2$ $\quad h(0-2) = h(-2)$

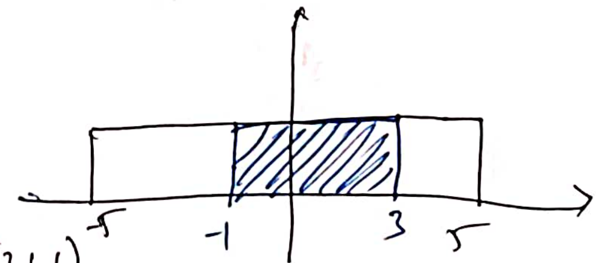


$$\int_{-2}^{2} A \cdot A \cdot dt = A^2[t]_{-2}^{2} = A^2[2-(-2)]$$
$$= 4A^2$$

$t=1$

$\tau=-2$ $\quad h(1+2) = h(3)$

$\tau = 2$ $\quad h(1-2) = h(-1)$

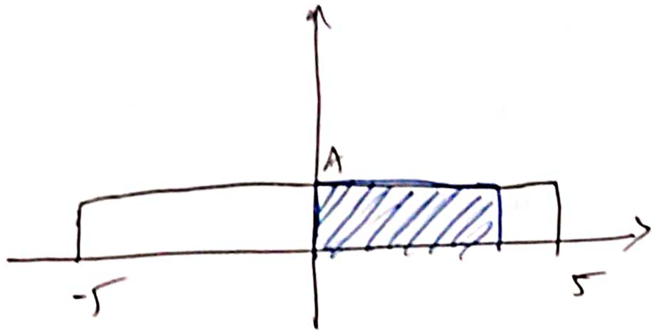


$$\int_{-1}^{3} A \cdot A \cdot dt = A^2[t]_{-1}^{3} = A^2[3+1]$$
$$= 4A^2$$

$t=2$

$\tau=-2 \quad h(t-\tau) = h(2+2) = h(4)$

$\tau=2 \quad h(2-2) = h(0).$

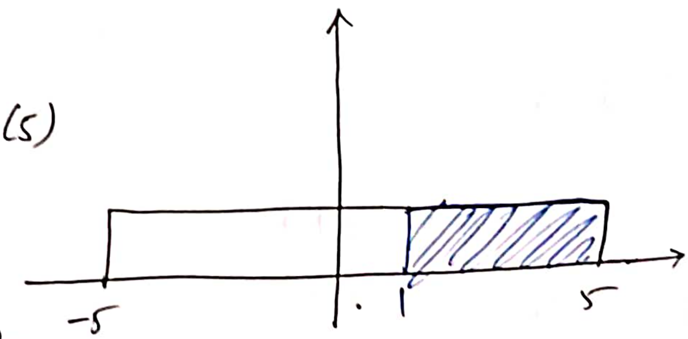


$$\int_0^4 A \cdot A \cdot dt = A^2 [t]_0^4 = \underline{\underline{4A^2}}$$

$t=3$

$\tau=-2 \quad h(t-\tau) = h(3-(-2)) = h(5)$

$\tau=2 \quad h(3-2) = h(1) = 1$

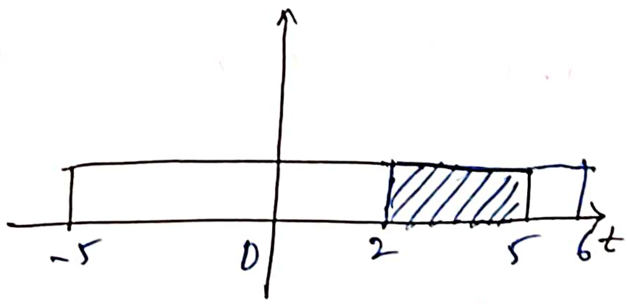


$$\int_1^5 A \cdot A \cdot dt = A^2 [t]_1^5 = \underline{\underline{4A^2}}$$

$t=4$

$\tau=-2 \quad h(t-\tau) = 4-(-2) = h(6)$

$\tau=2 \quad h(t-\tau) = 4-2 = h(2).$

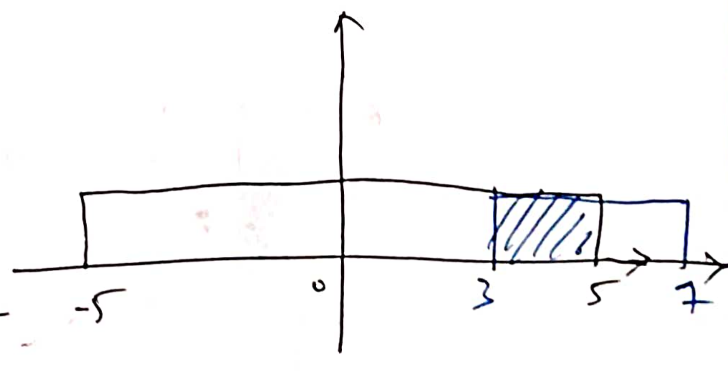


$$\int_2^5 A A \cdot dt = A^2 [t]_2^5 = \underline{\underline{3A^2}}$$

$t=5$

$\tau=-2 \quad h(t-\tau) = h(5-(-2)) = h(7)$

$\tau=2 \quad h(t-\tau) = h(5-2) = h(3)$



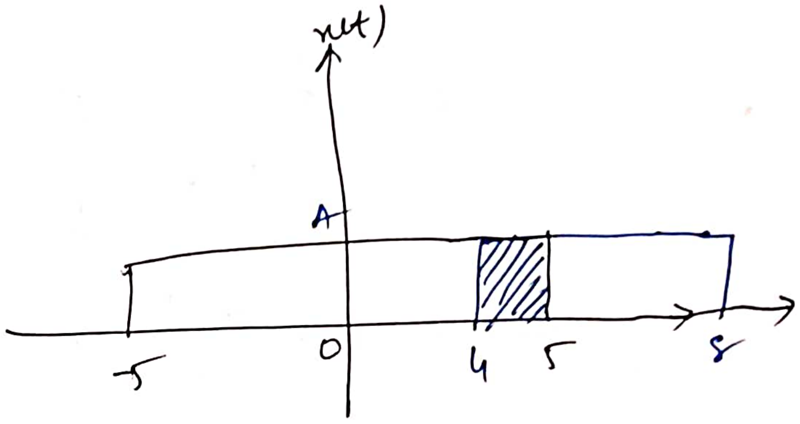
$$\int_3^5 A \cdot A \cdot dt = A^2 [t]_3^5 = \underline{\underline{2A^2}}$$

$t = 6$

$\tau = -2 \quad h(t-\tau) = h(6-(-2)) = h(+8)$

$h(6-2) = h(+4)$

$$\int_4^5 A \cdot A \cdot dt = A^2[t]_4^5 = A^2$$

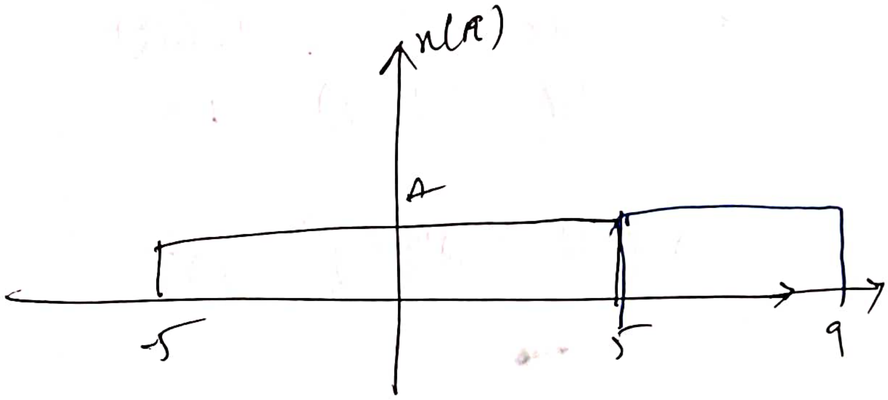


$t = 7$

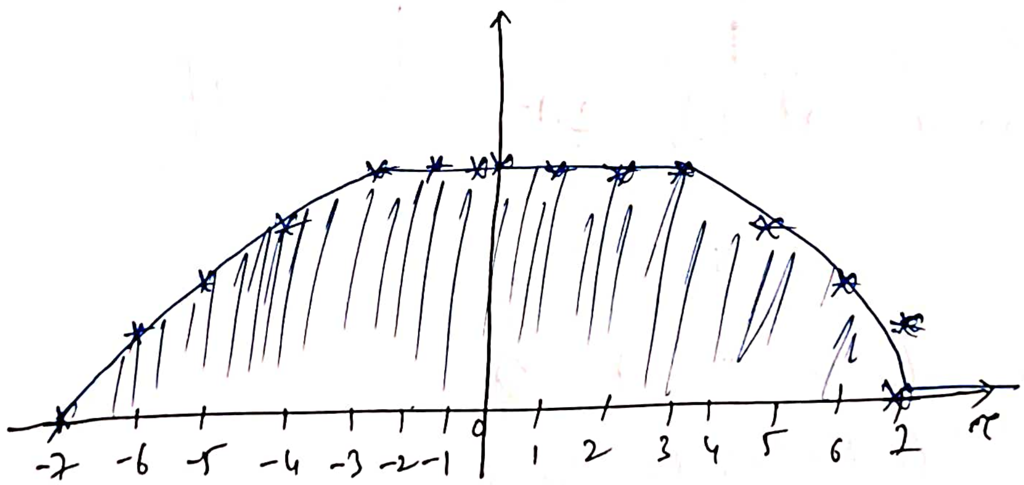
$\tau = -2 \quad h(t-\tau) = h(9)$

$\tau = 2 \quad (7-2)\ h(5)$

$= 0$



now the convolution of $x(\tau)$ and $h(t)$ can be plotted as



Laplace Transforms

Laplace Transform represents continuous time signals in terms of Complex exponentials i.e. e^{-st}. It is used to analyze the signals or functions which are not absolutely integrable

→ More effectively continuous time signals can be analyzed using Laplace Transforms.

→ Laplace transform provides broader characterization compared to F.T.

Definition

To transform a time domain signal $x(t)$ to s-domain, multiply the signal with e^{-st} and then integrate from $-\infty$ to ∞.

The transformed signal is represented as $X(s)$ and transformation is denoted by letter $\mathcal{L}$

$$X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt$$

where s is complex in nature and given as

$s = \sigma + j\omega$ $\quad$ $\sigma \rightarrow$ real part / attenuation constant

$j\omega \rightarrow$ imaginary part / complex frequency.

if $x(t)$ is defined for $t \geq 0$ then

$$\mathcal{L}\{x(t)\} = X(s) = \int_{0}^{\infty} x(t) \cdot e^{-st} \cdot dt$$

Bilateral L.T & unilateral L.T are two types in L.T.

$$\mathcal{L}^{-1}\{X(s)\} = x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) \cdot e^{st} \cdot ds$$

Relation B/W L.T and F.T

Fourier transform is given as

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt \quad \text{--- (1)}$$

F.T can be calculated only if $x(t)$ is absolutely integrable. i.e.

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

L.T

$$X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt \quad \text{by putting } s = \sigma + j\omega$$

$$= \int_{-\infty}^{\infty} x(t) \cdot e^{-(\sigma + j\omega)t} \cdot dt = \int_{-\infty}^{\infty} x(t) \cdot e^{-\sigma t} \cdot e^{-j\omega t} \cdot dt \quad \text{--- (2)}$$

by comparing the equations

F.T. of $x(t) e^{-\sigma t}$ is equal to L.T of $x(t)$.

if $\sigma = 0$ $\quad s = j\omega$

$$X(s) = \int_{-\infty}^{\infty} x(t) \, e^{-j\omega t} \cdot dt = X(j\omega) \text{ when } s = j\omega$$

It is basically F.T on imaginary axis in S-plane

Region of Convergence:

Condition for Existance of laplace transform:

$$X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot e^{-(\sigma+j\omega)t} \cdot dt = \int_{-\infty}^{\infty} x(t) \cdot e^{-\sigma t} \cdot e^{-j\omega t} \cdot dt$$

Condition for Existance of L.T

$$\boxed{\int_{-\infty}^{\infty} |x(t)\, e^{-\sigma t}|\, dt < \infty} \rightarrow \text{range of } \sigma = \text{ROC}$$

Find the ROC of signal $f(t) = e^{2t} u(t)$

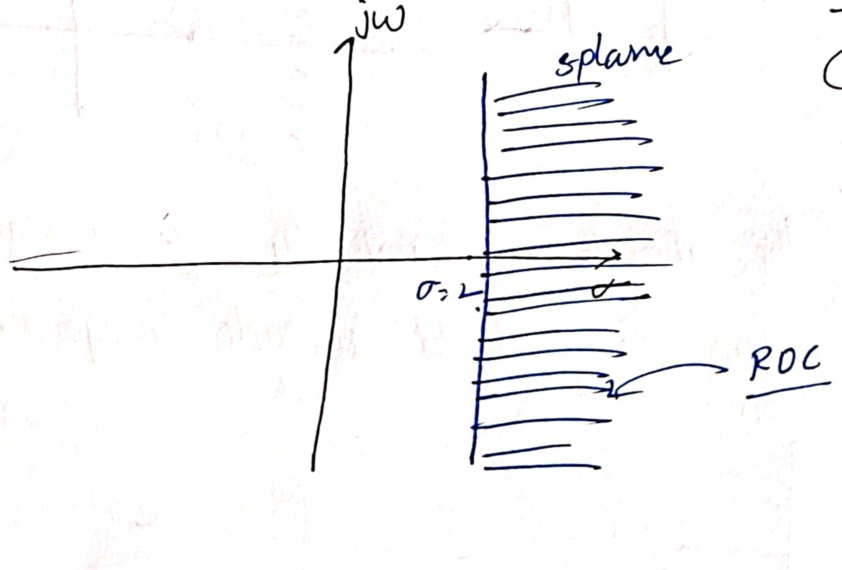
$$\int_{-\infty}^{\infty} |f(t)\, e^{-\sigma t}|\, dt = \int_{-\infty}^{\infty} |e^{2t} u(t) \cdot e^{-\sigma t}|\, dt$$

$$= \int_{0}^{\infty} |e^{(2-\sigma)t}|\, dt < \infty \quad \text{when } 2-\sigma < 0$$

$$-\sigma < -2$$

(or)

$$\sigma > 2$$

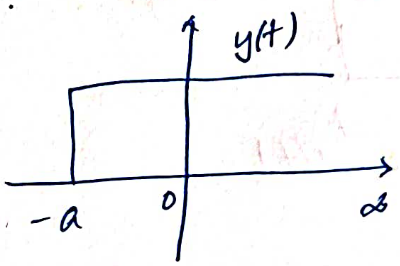


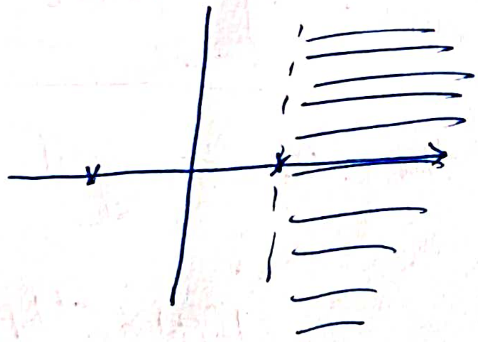
ROC: It is the range of complex variables 's' in s-plane for which laplace transform is finite or convergent.

$$x_1(t) \rightleftharpoons X_1(s) \qquad \boxed{X_1(s)} = \boxed{X_2(s)}$$
$$x_2(t) \rightleftharpoons X_2(s) \qquad \sigma > 2 \qquad \sigma < -1$$

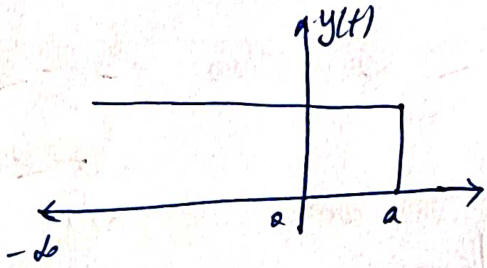
ROC properties

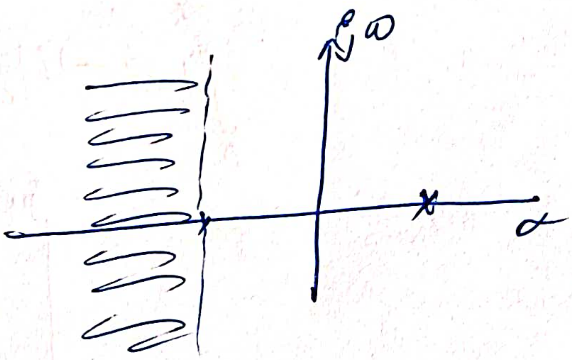
1) ROC does not include any poles
2) For right sided signals, ROC is right side to the rightmost pole.





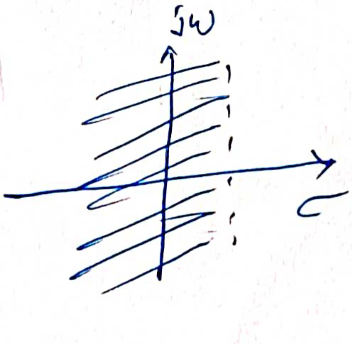
3) For left sided signals, ROC is left side to the leftmost pole

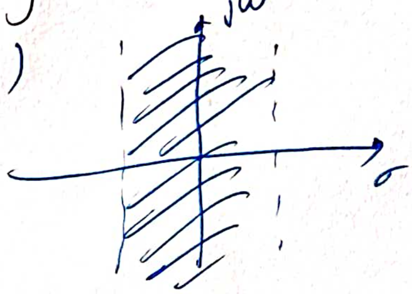




4) For the absolute integrability of a signal or the stability of a system. ROC should include imaginary axis.

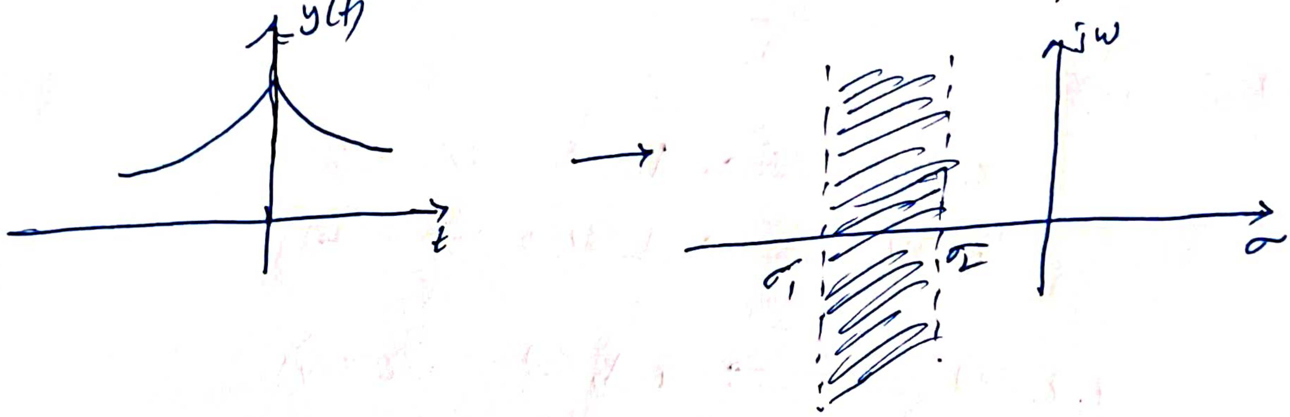
i)


ii)


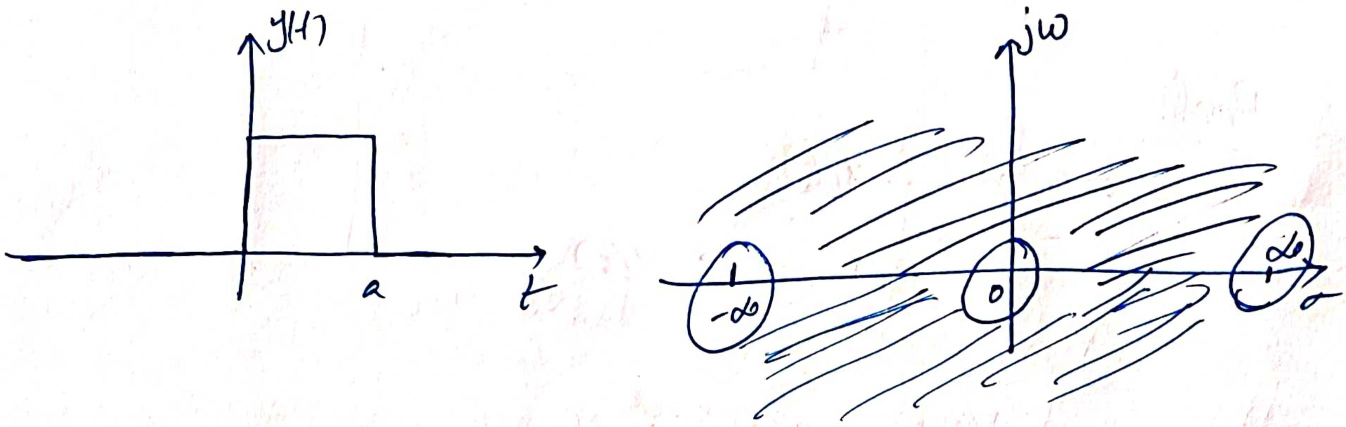
iii)


③

5. For both sided signals, ROC is a strip in the s-plane.



6. For finite duration signals, ROC is the entire s-plane excluding $s=0$ &/or $+\infty$ &/or $-\infty$.



Simple steps to calculate ROC

1) Compare σ with the real part of the coefficients of t in power of e.

2) Check if the signal is left sided or right sided and decide $\leq$ or $\geq$.

Qn. $x(t) = e^{(2+3j)t}\, u(-t-2)$

↳ LSS (Left sided signal)

$\boxed{\sigma < 2}$ ROC

Properties of Laplace Transform:

1) Linearity

$$x_1(t) \xrightleftharpoons{LT} X_1(s) \quad ROC = R_1$$

$$x_2(t) \xrightleftharpoons{LT} X_2(s) \quad ROC = R_2$$

$$a.x_1(t) \xrightleftharpoons{LT} a.X_1(s). \quad ROC = R_1$$

$$b.x_2(t) \xrightleftharpoons{LT} b.X_2(s). \quad ROC = R_2$$

$$a.x_1(t) + b.x_2(t) \xrightleftharpoons{L.T} a.X_1(s) + b.X_2(s) \quad \underline{ROC: R_1 \cap R_2}$$

Proof:

$$\underbrace{a.x_1(t) + b.x_2(t)}_{y(t)} \rightleftharpoons Y(s)$$

$$Y(s) = \int_{-\infty}^{\infty} y(t).e^{-st}.dt$$

$$= \int_{-\infty}^{\infty} (a.x_1(t) + b.x_2(t))e^{-st}.dt$$

$$= a\int_{-\infty}^{\infty} x_1(t).e^{-st}.dt + b\int_{-\infty}^{\infty} x_2(t).e^{-st}.dt$$

$$= \underline{a.X_1(s) + b.X_2(s)}$$

2: Conjugation:

$$x(t) \xrightleftharpoons{L.T} X(s)$$

$$x^*(t) \rightleftharpoons X^*(s^*)$$

Proof:

$$\cancel{X}(s) = \int_{-\infty}^{\infty} \cancel{x}(t) \cdot e^{-st} \cdot dt$$

$$X^*(s) = \int_{-\infty}^{\infty} x^*(t) \cdot e^{-s^*t} \cdot dt$$

replace s^* by s

$$X^*(s^*) = \int_{-\infty}^{\infty} x^*(t) \cdot e^{-st} \cdot dt$$

$$X^*(s^*) \xrightleftharpoons[LT]{ILT} x^*(t)$$

3. Time Reversal:

$$x(t) \xrightleftharpoons[ILT]{LT} X(s) \quad ROC = R$$

$$x(-t) \xrightleftharpoons{LT} \cancel{X}(-s) \quad ROC = -R$$

Proof:

$$\cancel{X}(s) = \int_{-\infty}^{\infty} x(t) e^{-st} \cdot dt$$

replace t by $-t$.

$$= \int_{-\infty}^{\infty} x(-t) \cdot e^{-st} \cdot dt.$$

put $-t = \tau$.

$$= \int_{-\infty}^{\infty} x(\tau) e^{+s\tau} \cdot d\tau$$

$$= \underbrace{\int_{-\infty}^{\infty} x(\tau) \cdot e^{-(-s)\tau} \cdot d\tau}_{X(-s)}$$

4 : Time Scaling

$$x(t) \rightleftharpoons X(s) \qquad ROC = R$$

$$x(at) \rightleftharpoons \frac{1}{|a|} X\left(\frac{s}{a}\right) \qquad ROC = |a|.R$$

5: Time shifting

$$x(t) \rightleftharpoons X(s) \qquad ROC = R$$

Right shift $\Leftarrow$ $x(t-t_0) \rightleftharpoons X(s).e^{-st_0} \qquad ROC = R$

Left shift $\leftarrow$ $x(t+t_0) \rightleftharpoons e^{st_0}.X(s). \qquad ROC = R$

6 : Frequency shifting

$$x(t) \xrightleftharpoons{LT} X(s) \qquad ROC = R$$

$$e^{s_0 t}.x(t) \rightleftharpoons X(s-s_0) \qquad ROC = R + Re\{s_0\}$$

$$e^{-s_0 t}.x(t) \rightleftharpoons X(s+s_0) \quad ROC = R + Re\{s_0\}$$

7. Convolution in time

$$x_1(t) \rightleftharpoons X_1(s) \quad ROC = R_1$$

$$x_2(t) \rightleftharpoons X_2(s) \quad ROC = R_2$$

$$x_1(t) \circledast x_2(t) \xrightleftharpoons{L.T} X_1(s).X_2(s) \quad ROC = R_1 \cap R_2$$

8. Multiplication in time

$$x_1(t).x_2(t) \rightleftharpoons \frac{1}{2\pi j}[X_1(s) * X_2(s)] \quad ROC = R_1 \cap R_2$$

9 differentiation in Time.

$$x(t) \rightleftharpoons X(s) \quad ROC = R$$

$$\left.\begin{array}{l} \frac{d}{dt}x(t) \rightleftharpoons s.X(s). \quad ROC = R \\ \frac{d^n}{dt^n}x(t) \rightleftharpoons s^n.X(s) \quad ROC = R \end{array}\right\} \text{Bilateral}$$

if unilateral

$$\frac{d^n x(t)}{dt^n} \rightleftharpoons s^n X(s) - s^{n-1} x(0^-) - s^{n-2} x'(0^-) - s^{n-3} x''(0^-) - - - -$$

$$x(0^-) = x(t)\Big|_{t=0^-}$$

$$x'(0^-) = \frac{d}{dt}x(t)\Big|_{t=0^-}$$

$$x''(0^-) = \frac{d^2}{dt^2}x(t)\Big|_{t=0^-}$$

10 Integration in time

$$x(t) \rightleftharpoons X(s), \quad ROC = R$$

$-\infty$ to t

$t \to \tau$

$$x'(t) = \int_{-\infty}^{\infty} x(\tau)\,d\tau \rightleftharpoons \frac{F(s)}{s}. \quad ROC = R \cap Re\{s\} > 0.$$

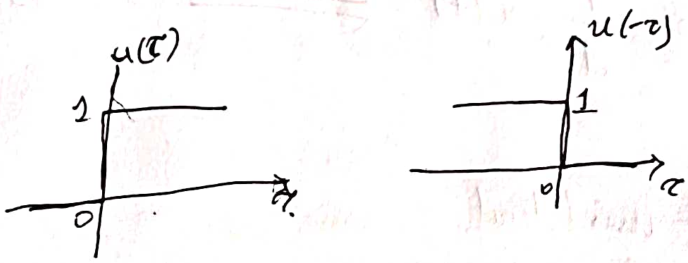
Bilateral Laplace Tran

$$= \int_{-\infty}^{t} x(\tau).d\tau \rightleftharpoons \frac{X(s)}{s} + \underline{\int_{-\infty}^{0^-} x(\tau)\,d\tau}$$

unilateral L.T.

Proof:

$$x(t) * u(t) \Longrightarrow \int_{-\infty}^{\infty} x(\tau) u(t-\tau)\,d\tau$$



$$= \int_{-\infty}^{t} x(\tau) 1 . d\tau + \int_{t}^{\infty} 0 . dt$$

$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau).d\tau \overset{LT}{\Longleftrightarrow} \frac{F(s)}{s}$$

Initial and Final Value Theorem

If ~~transform~~ $X(s)$ is a transform of unknown function $x(t)$. then it is possible to determine the initial value of $x(t)$.

i.e initial value of $x(t)$ at $t=0^+$.

Conditions 1: It is applicable only when $x(t)=0$ $t<0$

Condition 2: $x(t)$ must not contain any impulse or higher order singularities (discontinuties) at $t=0$.

Proof: $$\frac{d^n}{dt^n}x(t) \xrightleftharpoons[]{\text{unilateral signal}} s^n X(s) - s^{n-1}x(0^-) - s^{n-2}x'(0^-)$$

$$\frac{d}{dt}x(t) = s.X(s) - x(0^-)$$

$$s.X(s) - x(0^-) = \int_0^{\infty} \frac{d}{dt}x(t).e^{-st}.dt$$

$$= \int_{0^-}^{0^+} \frac{d}{dt}x(t).e^{-st}.dt + \int_{0^+}^{\infty} \frac{d}{dt}x(t).e^{-st}.dt$$

0^+ and 0^- are close to 0 so.

$$\simeq \int_{0^-}^{0^+} \frac{d}{dt}x(t).dt + \int_{0^+}^{\infty} \frac{d}{dt}x(t).e^{-st}.dt$$

$$s.X(0) - x(0^-) = x(0^+) - x(0^-) + \int_{0^+}^{\infty} \frac{d}{dt}x(t).e^{-st}.dt$$

$$s.X(s) = x(0^+) + \int_{0^+}^{\infty} \frac{dx}{dt}(t) \cdot e^{-st} \cdot dt$$

$$\lim_{s\to\infty} s.X(s) = x(0^+) + \int_{0^+}^{\infty} \frac{d}{dt}x(t) \lim_{s\to\infty} e^{-st} \cdot dt$$

$$\lim_{s\to\infty} s.X(s) = x(0^+) + 0$$

$$\boxed{x(0^+) = \lim_{s\to\infty} s.X(s)}$$

Final value Theorem

If $X(s)$ is a transform of unknown function / signal $x(t)$ then it is possible to determine the final value of $x(t)$. i.e the value of $x(t)$ at $t=\infty$.

Condition 1: It is applicable only when $x(t)=0$ where $t<0$.

Condition 2: $sF(s)$ must have poles in the left half of the s-plane.

Proof:

$$\frac{d}{dt}x(t) \Longleftrightarrow s.X(s) - x(0^-)$$

$$s X(s) - x(0^-) = \int_0^{\infty} \frac{d\,x(t)}{dt} \cdot e^{-st} \cdot dt$$

$$\lim_{s\to 0} s.X(s) - \lim_{s\to 0} x(0^-) = \int_0^{\infty} \frac{d}{dt}x(t) \cdot \lim_{s\to 0} e^{-st} \cdot dt$$

$$\lim_{s\to 0} s.X(s) - x(0^-) = \int_0^{\infty} \frac{d}{dt}x(t)\, dt$$

$$\lim_{s\to 0} s.X(s) - x(0^-) = [x(t)]_0^\infty$$

$$\lim_{s\to 0} s.X(s) - x(0^-) = x(\infty) - x(0)$$

$\therefore$

$$\boxed{x(\infty) = \lim_{s\to 0} s.X(s)}$$

Problems:

1. find the LT of $\delta(t)$.

Sol

$$X(s) = \int_{-\infty}^{\infty} x(t).e^{-st}.dt$$

$$= \int_{-\infty}^{\infty} \delta(t).e^{-st}.dt = \int_{-\infty}^{\infty} \delta(t-0).e^{-st}.dt.$$

$$= \delta(t-0).e^0 = \delta(t)$$

$$= \underline{1}$$ Entire s-plane will be ROC

2. L.T of unit step signal.

Sol.

$$X(s) = \int_{-\infty}^{\infty} x(t).e^{-st}.dt$$

$$= \int_{-\infty}^{0} 0.e^{-st}.dt + \int_{0}^{\infty} 1.e^{-st}.dt$$

$$= 0 + \left.\frac{e^{-st}}{-s}\right]_0^\infty = \frac{e^{-\infty}}{-s} - \frac{e^0}{-s}$$

$$= 1/s$$ ROC is $Re\{s\} > 0$

3 Find the laplace Transform of

(a) $x(t) = -e^{-at} u(-t)$

(b) $x(t) = e^{at} u(-t)$

Sol:

$$x(t) = -e^{-at} u(-t)$$

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt$$

$$= -\int_{-\infty}^{+\infty} e^{-at} u(-t) \cdot e^{-st} \cdot dt = -\int_{-\infty}^{0^-} e^{-at} \cdot e^{-st} \cdot dt$$

$$= -\int_{-\infty}^{0^-} e^{-(s+a)t} \cdot dt = \frac{1}{s+a} e^{-(s+a)t} \Big|_{-\infty}^{0^-} = \frac{1}{s+a}$$

$$\boxed{Re(s) < -a}$$

$$-e^{-at} u(-t) \longleftrightarrow \frac{1}{s+a} \qquad Re(s) < -a$$

b) lly $X(s) = \int_{-\infty}^{\infty} e^{at} u(-t)\, e^{-st} \cdot dt = \int_{-\infty}^{0^-} e^{at} \cdot e^{-st} \cdot dt$

$$= \int_{-\infty}^{0^-} e^{-(s-a)t} \cdot dt = \frac{1}{-(s-a)} \Big|_{-\infty}^{0^-} = \frac{-1}{s-a} \qquad Re(s) < a$$

4. Find the Laplace transform of $x(t)$. where $x(t) = t.u(t)$. ⑧

Sol:

$$x(t) = t \qquad t \geq 0$$
$$= 0 \qquad t < 0$$

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t).e^{-st}.dt$$

$$= \int_0^{\infty} t.e^{-st}.dt = \left[t.\frac{e^{-st}}{-s}\right]_0^{\infty} - \int_0^{\infty} 1.\frac{e^{-st}}{-s}dt$$

$$= \left[t.\frac{e^{-st}}{-s}\right]_0^{\infty} - \left[\frac{e^{-st}}{s^2}\right]_0^{\infty}$$

$$= \infty.\frac{e^{-s\infty}}{-s} - 0 - \frac{e^{-s\infty}}{s^2} + \frac{1}{s^2}$$

$$= \frac{1}{s^2} + \infty.\frac{e^{-(\sigma+j\omega)\infty}}{-s} - \frac{e^{-(\sigma+j\omega)\infty}}{s^2}$$

for +ve values of σ $x(t)$. converges

$$= \frac{1}{s^2}$$ ROC is right half of the s-plane.

5 Find the L.T of following equation

$$x(t) = e^{-4|t|}$$

$$= e^{-4t} \quad \text{for } t \geq 0$$

$$= e^{4t} \quad \text{for } t < 0$$

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt$$

$$= \int_{-\infty}^{0} e^{4t} \cdot e^{-st} \cdot dt + \int_{0}^{\infty} e^{-4t} \cdot e^{-st} \cdot dt$$

$$= \int_{-\infty}^{0} e^{-(s-4)t} \cdot dt + \int_{0}^{\infty} e^{-(s+4)t} \cdot dt$$

$$= \left[\frac{e^{-(s+4)t}}{-(s-4)} \right]_{-\infty}^{0} + \left[\frac{e^{-(s+4)}}{-(s+4)} \right]_{0}^{\infty}$$

$$= \left[\frac{1}{-(s-4)} + \frac{e^{(s-4)\infty}}{s-4} - \frac{e^{-(s+4)\infty}}{s+4} + \frac{1}{s+4} \right]$$

$$= \left[\frac{-1}{s-4} - \frac{1}{s+4} \right) = \frac{-8}{s^2-16}$$

6 Determine the L.T of following signals ①

(i) $x(t) = \sin\omega_0 t \cdot u(t)$.

$x(t) = \sin\omega_0 t$ for $t \geq 0$

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}.dt = \int_0^{\infty} \sin\omega_0 t.\, e^{-st}.dt$$

$$\sin\omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

So $$= \int_0^{\infty} \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \cdot e^{-st}.dt$$

$$= \frac{1}{2j}\int_0^{\infty}\left(e^{-(s-j\omega_0)t} - e^{-(s+j\omega_0)t}\right)dt$$

$$= \frac{1}{2j}\left[\frac{e^{-(s-j\omega_0)t}}{-(s-j\omega_0)} - \frac{e^{-(s+j\omega_0)t}}{-(s+j\omega_0)}\right]_0^{\infty}$$

$$= \frac{1}{2j}\left[\frac{+1}{s-j\omega_0} - \frac{1}{s+j\omega_0}\right]$$

$$= \frac{1}{2j}\left[\frac{s+j\omega_0 - s + j\omega_0}{s^2+\omega_0^2}\right]$$

$$= \frac{2j\omega_0}{2j(s^2+\omega_0^2)} = \frac{\omega_0}{s^2+\omega_0^2}$$

ii) $x(t) = \cos\omega_0 t \cdot u(t)$

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt \qquad \cos\omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$= \int_{-\infty}^{\infty} \cos\omega_0 t \cdot u(t) \cdot e^{-st} \cdot dt$$

$$= \int_{0}^{\infty} \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \cdot e^{-st} \cdot dt = \int_{0}^{\infty} \frac{e^{-(s-j\omega_0)t} + e^{-(s+j\omega_0)t}}{2} \cdot dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-j\omega_0)t}}{-(s-j\omega_0)} + \frac{e^{-(s+j\omega_0)t}}{-(s+j\omega_0)}\right]_0^{\infty}$$

$$= \frac{1}{2}\left[\frac{1}{s-j\omega_0} + \frac{1}{s+j\omega_0}\right] = \frac{s+j\omega_0+s-j\omega_0}{2(s^2+\omega_0^2)}$$

$$= \frac{\cancel{2}s}{\cancel{2}(s^2+\omega_0^2)} = \underline{\underline{\frac{s}{s^2+\omega_0^2}}}$$

(iii) $x(t) = e^{-at} \cdot \sin\omega_0 t \cdot u(t)$

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt$$

$$= \int_{0}^{\infty} e^{-at} \cdot \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \cdot e^{-st} \cdot dt$$

$$= \frac{1}{2j}\left[\int_0^{\infty} e^{-(s+a-j\omega_0)t} - e^{-(s+a+j\omega_0)t} \cdot dt\right]$$

$$= \frac{1}{2j}\left[\frac{e^{-(s+a-j\omega)t}}{-(s+a-j\omega_0)} - \frac{e^{-(s+a+j\omega_0)t}}{-(s+a+j\omega_0)}\right]_0^{\infty}$$

$$= \frac{1}{2j}\left[\frac{e^{-\infty}}{-(s+a-j\omega)} + \frac{e^{0}}{s+a-j\omega_0} + \frac{e^{-\infty}}{(s+a+j\omega)} - \frac{e^{0}}{s+a+j\omega_0}\right]$$

$$= \frac{1}{2j}\left[\frac{1}{s+a-j\omega_0} + \frac{1}{s+a+j\omega_0}\right] = \frac{1}{2j}\left[\frac{(\cancel{s}+\cancel{a}+j\omega_0 - \cancel{s}\cancel{+a}+j\omega_0)}{(s+a)^2+\omega_0^2}\right]$$

$$= \frac{1}{\cancel{2j}}\,\frac{\cancel{2j}\omega_0}{(s+a)^2+\omega_0^2} = \frac{\omega_0}{(s+a)^2+\omega_0^2}.$$

(iv) $x(t) = e^{-at}\cos\omega_0 t \cdot u(t)$

Sol: $x(t) = e^{-at}\cos\omega_0 t$ for $t \geq 0$

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt$$

$$= \int_{-\infty}^{\infty} e^{-at} \cdot \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} u(t) e^{-st} \cdot dt = \int_0^{\infty} \frac{e^{-(s+a-j\omega_0)t} + e^{-(s+a+j\omega_0)t}}{2} \cdot dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s+a-j\omega_0)t}}{-(s+a-j\omega_0)} + \frac{e^{-(s+a+j\omega_0)t}}{-(s+a+j\omega_0)}\right]_0^\infty$$

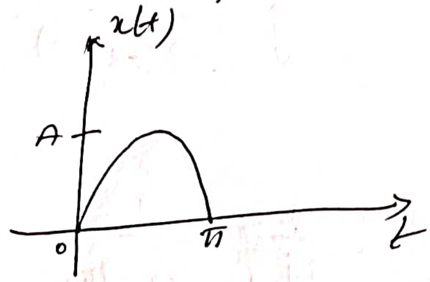
$$= \frac{1}{2}\left[\frac{s+a+j\omega_0 + s+a-j\omega_0}{(s+a)^2+\omega_0^2}\right] = \frac{s+a}{(s+a)^2+\omega_0^2}$$

7. Determine the laplace transform of sine pulse.

Sol:

$x(t) = A\sin t$ for $0<t<\pi$

$= 0$ for $t>\pi$



$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\cdot e^{-st}\cdot dt$$

$$= \int_0^{\pi} A\sin t \cdot e^{-st}\,dt = \int_0^{\pi} A\cdot\frac{e^{jt}-e^{-jt}}{2j}\cdot e^{-st}\cdot dt$$

$$= \frac{A}{2j}\int_0^{\pi}\left(e^{-(s-j)t} - e^{-(s+j)t}\right)\cdot dt$$

$$= \frac{A}{2j}\left[\frac{e^{-(s-j)t}}{-(s-j)} - \frac{e^{-(s+j)t}}{-(s+j)}\right]_0^{\pi} = \frac{A}{2j}\left[\frac{e^{-(s-j)\pi}-e^{0}}{-(s-j)} - \frac{e^{-(s+j)\pi}-e^{0}}{-(s+j)}\right]$$

$$= \frac{A}{2j}\left[\frac{e^{-(s-j)\pi}-1}{-(s-j)} - \frac{e^{-(s+j)\pi}-1}{-(s+j)}\right]$$

(11)

$$= \frac{A}{2j}\left[\frac{e^{-(s-j)\pi}}{-(s-j)} - \frac{e^{-(s+j)\pi}}{-(s+j)} + \frac{1}{s-j} - \frac{1}{s+j}\right]$$

$$= \frac{A}{2j}\left[\frac{e^{-(s+j)\pi}}{s+j} - \frac{e^{-(s-j)\pi}}{s-j} + \frac{1}{s-j} - \frac{1}{s+j}\right]$$

$$= \frac{A}{2j}\left[\frac{e^{-(s+j)\pi}\cdot(s-j) - e^{-(s-j)\pi}\cdot(s+j) + \cancel{s}+j - \cancel{s}+j}{s^2+1}\right]$$

$$= \frac{A}{2j}\left[\frac{s\cdot e^{-s\pi}\cdot e^{-j\pi} - j\cdot e^{-s\pi}\cdot e^{-j\pi} - s e^{-s\pi}\cdot e^{+j\pi} - j e^{-s\pi}\cdot e^{j\pi} + 2j}{s^2+1}\right]$$

$$= \frac{A}{2j}\left[\frac{s\cdot e^{-s\pi}\left(e^{-j\pi} - e^{j\pi}\right] - j e^{-s\pi}\left[e^{-j\pi} - e^{j\pi}\right] + 2j}{s^2+1}\right]$$

$$= \frac{A}{2j}\cdot\frac{0 + 2j e^{-s\pi} + 2j}{s^2+1} = \frac{A}{\cancel{2j}}\frac{\cancel{2j}\left(e^{-s\pi}+1\right)}{s^2+1}$$

$$= \frac{A\cdot\left(e^{-s\pi}+1\right)}{s^2+1}$$

8. Find the inverse laplace transform of the following $X(s)$

a) $X(s) = \dfrac{2s+4}{s^2+4s+3}$, $Re(s) > -1$

b) $X(s) = \dfrac{2s+4}{s^2+4s+3}$, $Re\{s\} < -3$

c) $X(s) = \dfrac{2s+4}{s^2+4s+3}$, $-3 < Re(s) < -1$

Sol,

$$X(s) = \frac{2s+4}{s^2+4s+3} = \frac{2.\ s+2}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$$

Multiply $(s+1)(s+3)$ on both sides

$$2.(s+2) = A(s+3) + B(s+1)$$

if $s = -1$

$$2.(-1+2) = A(-1+3) + B(-1+1)$$

$$A = \frac{2}{2} = 1$$

if $s = -3$

$$2(-3+2) = A(-3+3) + B(-3+1)$$

$$-2B = -2$$

$$B = 1$$

$$X(s) = \frac{1}{s+1} + \frac{1}{s+3}$$

a) The ROC of $X(s)$ is $Re(s) > -1$ Thus $x(t)$ is a right-sided signal so.

$$x(t) = e^{-t}u(t) + e^{-3t}u(t).$$

$$= (e^{-t} + e^{-3t})\, u(t)$$

b) The ROC of $X(s)$ is $Re(s) < -3$. Thus $x(t)$ is a left-sided signal and

$$x(t) = -e^{-t}u(-t) - e^{-3t}u(-t)$$

$$= -(e^{-t} + e^{-3t})u(-t)$$

c) The ROC of $X(s)$ is $-3 < Re(s) < -1$, thus $x(t)$ ~~has a~~ strip ROC. So

$$x(t) = -e^{-t}u(-t) + e^{-3t}u(t)$$

9 Find the inverse laplace transform of

$$X(s) = \frac{5s+13}{s(s^2+4s+13)} \qquad Re(s) > 0$$

Sol.

$$X(s) = \frac{5s+13}{s(s^2+4s+13)} = \frac{A}{s} + \frac{Bs+C}{s^2+4s+13}$$

$$5s+13 = A(s^2+4s+13) + (Bs+C)\cdot s$$

$$As^2 + Bs^2 + 4As + Cs + 13A = 5s + 13$$

equating the common power

$$A+B = 0 \quad — (1)$$
$$4A+C = 5 \quad — (2)$$
$$13A = 13 \quad — (3)$$

$$A = \frac{\cancel{13}}{\cancel{13}} = 1$$

$$A+B = 0$$
$$B = -A = -1$$

$$4A+C = 5$$
$$4+C = 5$$
$$C = 1$$

$$X(s) = \frac{1}{s} + \frac{-s+1}{s^2+4s+9}$$

$$= \frac{1}{s} - \frac{s+2-3}{(s^2+4s+4+9)} = \frac{1}{s} - \frac{s+2-3}{(s+2)^2+3^2}$$

$$\frac{s+2}{(s+2)^2+(3)^2} - \frac{3}{(s+2)^2+3^2}$$

$$X(s) = \frac{1}{s} + \frac{s+2}{(s+2)^2+(3)^2} + \frac{3}{(s+2)^2+(3)^2}$$

$$x(t) = u(t) - e^{-2t}\cos 3t\, u(t) + e^{-2t}\sin 3t\, u(t)$$

10. Find the inverse laplace transforms of $X(s) = \frac{s^2+2s+5}{(s+3)(s+5)^2}$

$Re(s) > -3$

Sol:

$$X(s) = \frac{s^2+2s+5}{(s+3)(s+5)^2} = \frac{A}{s+3} + \frac{B}{(s+5)} + \frac{C}{(s+5)^2}$$

$$s^2+2s+5 = A(s+5)^2 + B(s+5)(s+3) + C(s+3)$$

$$s^2+2s+5 = A(s^2+10s+25) + B(s^2+8s+15) + C(s+3)$$

equating like terms

$$A+B = 1$$

$$10A + 8B + C = 2$$

$$15B + 25A + 3C = 5$$

Substitute $s = -3$ in above equation

$$9 - 6 + 5 = A(4)$$

$$8 = 4A \longrightarrow A = 8/4 = 2$$

$$A + B = 1$$

$$B = 1 - A = 1 - 2 = -1$$

$$10A + 8B + C = 2$$

$$20 + 8(-1) + C = 2$$

$$20 - 8 + C = 2$$

$$C = -10$$

$$X(s) = \frac{2}{s+3} - \frac{1}{s+5} - \frac{10}{(s+5)^2}$$

$$= 2 \cdot e^{-3t} \cdot u(t) - 1 e^{-5t} u(t) - 10t \cdot e^{-5t} \cdot u(t)$$

12 Find the inverse Laplace transform of the following

a) $X(s) = \frac{2s+1}{s+2}$, $Re(s) > -2$

b) $X(s) = \frac{s^2+6s+7}{s^2+3s+2}$, $Re(s) > -1$

c) $X(s) = \frac{s^3+2s^2+6}{s^2+3s}$, $Re(s) > 0$

Sol,

a) $X(s) = \frac{2s+1}{s+2} = \frac{2s+1+3-3}{s+2}$

$$= \frac{2(s+2)-3}{s+2} = \frac{2\cancel{(s+2)}}{\cancel{s+2}} - \frac{3}{s+2}$$

$$= 2 - \frac{3}{s+2}$$

by apply ILT

$$= 2\delta(t) - 3 \cdot e^{-2t} \cdot u(t)$$

5) $X(s) = \dfrac{s^2+6s+7}{s^2+3s+2} = \dfrac{s^2+3s+2+3s+5}{s^2+3s+2}$

$$= \frac{s^2+3s+2}{s^2+3s+2} + \frac{3s+5}{(s+1)(s+2)}$$

$$= 1 + \frac{3s+5}{(s+1)(s+2)} \longrightarrow \frac{A}{s+1} + \frac{B}{s+2}$$

$$3s+5 = A(s+2) + B(s+1)$$

if $s = -2$

$$-6+5 = B(-2+1)$$

$$-1 = -B$$

$$\underline{\underline{B = 1}}$$

if $s = -1$

$$-3+5 = A(-1+2)$$

$$\underline{\underline{A = 2}}$$

$$= 1 + \frac{2}{s+1} + \frac{1}{s+2}$$

ILT on bothsides

$$= \delta(t) + 2 \cdot e^{-t} u(t) + 1 \cdot e^{-2t} u(t)$$

$$x(t) = \underline{\underline{\delta(t) + (2e^{-t} + e^{-2t}) u(t)}}$$

(c)

$$X(s) = \frac{s^3+2s^2+6}{s^2+3s} = \frac{s^3+2s^2+s^2-s^2+6}{s^2+3s}$$

$$= \frac{s^2(s+3) - s^2+6}{s^2+3s}$$

$$= \frac{s^2(s+3)}{s^2+3s} - \frac{s^2+6}{s^2+3s}$$

$$= \frac{s.\cancel{s}\cancel{(s+3)}}{\cancel{s(s+3)}} + \frac{s^2-6}{s^2+3s}$$

$$= s + \frac{s^2+3s-3s-6}{s^2+3s}$$

$$= s + \frac{\cancel{s^2+3s}}{\cancel{s^2+3s}} + \frac{3(s+2)}{s^2+3s}$$

$$= s - 1 + \frac{3(s+2)}{s^2+3s} \longrightarrow \frac{A}{s} + \frac{B}{s+3}$$

$$3(s+2) = A(s+3) + B.s$$

Put $s=0$

$$6 = 3A$$

$$\underline{A = 2}$$

Put $s=-3$

$$-3B = -3$$

$$\underline{B = 1}$$

$$= s - 1 + \frac{2}{s} + \frac{1}{s+3}$$

$$= \delta'(t) - \delta(t) + 2.u(t) + e^{-3t}.u(t)$$

$$x(t) = \underline{\underline{\delta'(t) - \delta(t) + (2+e^{-3t})u(t)}}$$

Z- Transform :→

CTFT = DTFT

LT = ZT

$$x[n] \underset{IZT}{\overset{ZT}{\rightleftharpoons}} X[z]$$

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] . z^{-n}$$

n → integer

z → Complex variable.

$\underbrace{\qquad}$ Bidirectional z- Transform

$z = r e^{j\omega}$ → phase or angle of z

r → magnitude of z

1) Bidirectional z - Transform

2) Unidirectional z - Transform

$$X[z] = \sum_{n=0}^{\infty} x[n]. z^{-n}$$

Ex:

$$x[n] = a^n . u[n] \rightleftharpoons X[z] = ?$$

$$X[z] = \sum_{n=-\infty}^{\infty} a^n u[n] . z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n . z^{-n}$$

$$= \sum_{n=0}^{\infty} (a z^{-1})^n$$

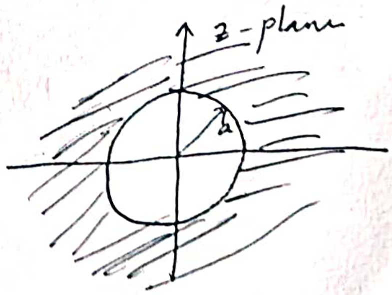
$$= 1 + a z^{-1} + (a z^{-1})^2 + \text{------} + (a z^{-1})^n + \text{----}$$

Common ratio $= a z^{-1}$

$$X[z] = \frac{1}{1 - a z^{-1}}, \quad |a z^{-1}| < 1$$

$$\left|\frac{a}{z}\right| < 1$$

$$r = \underbrace{|z| > a}_{ROC}$$



inside the circle z-transform doesnot exist.

Condition for Existance of z-transform:

$$x[n] \Longleftrightarrow X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}$$

$$|X(z)| < \infty$$

$$\left| \sum_{n=-\infty}^{\infty} x(n) z^{-n} \right| < \infty$$

$$\sum_{n=-\infty}^{\infty} |x(n) z^{-n}| < \infty$$

we know $z = re^{j\omega}$

$$\sum_{n=-\infty}^{\infty} |x[n] \cdot (re^{j\omega})^{-n}| < \infty$$

we know $|e^{-j\omega n}| = 1$

$$\sum_{n=-\infty}^{\infty} |x[n] \cdot r^{-n}| |e^{-j\omega n}| < \infty$$

$$\boxed{\sum_{n=-\infty}^{\infty} |x[n] r^{-n}| < \infty} \rightarrow \text{Existance of z-transform.}$$

if $r = 1$

$$\boxed{\sum_{n=-\infty}^{\infty} |x[n]| < \infty} \rightarrow \text{Existance of DTFT}$$

Eg: $x[n] = 2^n u[n]$, ROC = ?

$$\sum_{n=-\infty}^{\infty} |2^n u[n] r^{-n}| = \sum_{n=0}^{\infty} |2^n r^{-n}|$$

$$= 1 + 2r^{-1} + (2r^{-1})^2 + \cdots$$

$$= \frac{1}{1 - 2r^{-1}} \qquad |2r^{-1}| < 1$$

$$\left|\frac{2}{r}\right| < 1 \quad |z| > 2 \text{ (or) } r > 2$$

Properties of ROC

1) The ROC is a ring or disk in the z-plane centered at origin
2) The ROC cannot contain any poles
3) If $x(n)$ is a finite duration, causal sequence then the ROC is the entire z-plane except at $z=0$
4) If $x(n)$ is a finite duration, anti-causal sequence then the ROC is the entire z-plane except at $z=\infty$
5) If $x(n)$ is finite duration, two sided sequence then the ROC is entire z-plane except at $z=0$ & $z=\infty$
6) If $x(n)$ is an infinite duration, two sided sequence the ROC will consist of a ring in z-plane, bounded on interior and exterior by a plot not containing any poles.
7) The ROC of an LTI stable system contains unit circle.
8) The ROC must be connected region

Properties of z-tranform

1) Linearity

$$x_1[n] \rightleftharpoons X_1(z) \quad ROC = R_1$$

$$x_2[n] \rightleftharpoons X_2(z) \quad ROC = R_2$$

$$a \cdot x_1[n] + b \cdot x_2[n] \rightleftharpoons a \cdot X_1(z) + b \cdot X_2(z)$$

$$ROC \geq R_1 \cap R_2$$

2) Time shifting

$$x[n] \rightleftharpoons X(z) \quad ROC = R$$

$$x[n-n_0] \rightleftharpoons z^{-n_0} \cdot X(z)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x[n-n_0] \cdot z^{-n}$$

$$n - n_0 = m$$

$$n = m + n_0$$

$$= \sum_{m=-\infty}^{\infty} x[m] \cdot z^{-m} \cdot z^{-n_0}$$

$$= X(z) \cdot z^{-n_0}$$

3) Time Scaling

$x[n] \rightleftharpoons X(z)$, ROC $= R$

$x[n/m] \rightleftharpoons X[z^m]$ ROC $= R^{1/M}$

Proof

$x[n] \longrightarrow X(z)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-m}$$

$$X'(z) = \sum_{n=-\infty}^{\infty} x[\tfrac{n}{m}]\, z^{-n}$$

$n = km$

$$= \sum_{k=-\infty}^{\infty} x[k]\, z^{-km}$$

$$= \sum_{k=-\infty}^{\infty} x[k] (z^{m})^{-k} = X(z^m)$$

4 Time reversal

$x[n] \rightleftharpoons X[z]$, ROC $= R$

$x[-n] \rightleftharpoons X[z^{-1}]$ ROC $= R^{-1} = 1/R$

5 Scaling of z

$x[n] \rightleftharpoons X[z]$ ROC $= R$

$a^n x[n] \rightleftharpoons X[\tfrac{z}{a}]$ ROC $= |a|$

Proof

$$X(z) = \sum_{n=-\infty}^{\infty} a^n x[n] . z^{-n} = \sum_{n=-\infty}^{\infty} x[n] (a^{-1} z)^{-n}$$

$$X'(z) = X\left(\frac{z}{a}\right)$$

6. **Conjugation**

$$x[n] \rightleftharpoons X[z] \quad , \quad ROC = R$$

$$x^*[n] \rightleftharpoons X^*[z^*] \quad , \quad ROC = R$$

Proof

$$x[n] \rightleftharpoons X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$x^*[n] \rightleftharpoons \left[\sum_{n=-\infty}^{\infty} x[n] z^{-n}\right]^*$$

$$\sum_{n=-\infty}^{\infty} x^*[n] [z^*]^{-n}$$

$$X^*[z^*]$$

7. **Accumulation** ~~of~~ (Integration in time)

$x[n] \quad n \to k \quad x[k]$

$$\sum_{k=-\infty}^{n} x[k] \rightleftharpoons \frac{X(z)}{1-z^{-1}} \qquad ROC \quad R \cap |z| > 1$$

Proof $\quad x[n] * u[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot u[n-k]$

$$ZT\{x[n] * u(n)\} = \sum_{k=-\infty}^{n} x[k]$$

$$= ZT\{x(n)\} \, ZT\{u(n)\}$$

$$\boxed{ZT\left\{\sum_{k=-\infty}^{n} x[k]\right\} = X[z] \cdot \frac{1}{1-z^{-1}}} \qquad ROC \geq R \cap |z| > 1$$

8 Convolution

$$x_1[n] \rightleftharpoons X_1(z) \qquad ROC = R_1$$

$$x_2[n] \rightleftharpoons X_2(z) \qquad ROC = R_2$$

$$x_1[n] * x_2[n] \rightleftharpoons X_1[z] \cdot X_2[z] \qquad ROC \geq R_1 \cap R_2$$

Proof:

$$x_1[n] * x_2[n] \rightleftharpoons X'(z)$$

$$X'(z) = \sum_{n=-\infty}^{\infty} (x_1[n] * x_2[n]) \cdot z^{-n}$$

$$\sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1[k] \, x_2[n-k] \, z^{-n}$$

$$n-k = m \qquad n = k+m$$

$$\sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1(k) \cdot x_2[m] \cdot z^{-(k+m)}$$

$$\sum_{m=-\infty}^{\infty} x_2(m) \cdot z^{-m} \cdot \sum_{k=-\infty}^{\infty} x_1(k) \cdot z^{-k}$$

$$X_1(z) \cdot X_2(z)$$

9 Multiplication

$$x_1[n] \cdot x_2[n] \rightleftharpoons \frac{1}{2\pi j} \{ X_1(z) * X_2(z) \}$$

10 Differentiation

↓

successive difference (or) First difference property

$$\frac{d\,x[n]}{dn} = \frac{x[n] - x[n-1]}{n-(n-1)} = x[n] - x[n-1]$$

$$x[n] \xrightleftharpoons{Z.T} X(z) \quad ROC = R$$

$$X'(z) = \sum_{n=-\infty}^{\infty} (x[n] - x[n-1])\, z^{-n} = \sum_{n=-\infty}^{\infty} x[n].z^{-n} - \sum_{n=-\infty}^{\infty} x[n-1].z^{-n}$$

$$= (1-z^{-1})\, X(z)$$

$$= \left(\frac{z-1}{z}\right) X(z). \quad ROC = R^{*}$$

11 Differentiation in Z-domain

$$x[n] \rightleftharpoons X(z) \qquad ROC = R$$

$$n.x[n] \rightleftharpoons -z \frac{d}{dz} X[z] \qquad ROC = R$$

Proof

$$x[n] \rightleftharpoons X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$\frac{d}{dz} X[z] = \sum_{n=-\infty}^{\infty} x[n] \frac{dz^{-n}}{dz}$$

$$= \sum_{n=-\infty}^{\infty} n.x[n].\cancel{\frac{d}{dz}}\, z^{-n-1}$$

$$= -\sum_{n=-\infty}^{\infty} n.x[n] z^{-n}.z^{-1}$$

$$-z\frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} n.x[n].z^{-n}$$

$$\rightleftharpoons n.x[n]$$

(5)

12 **Initial value Theorem**

Conditions
1) $x[n] = 0, \; n < 0$

$$x[n] \rightleftharpoons X[z]$$

$$x[n]\big|_{n=0} = x[0] = \lim_{z \to \infty} X[z]$$

Proof

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n} = \sum_{n=0}^{\infty} x[n] \cdot z^{-n}$$

$$= x[0] + x[1] z^{-1} + \cdots$$

take limit on both sides

$$= \lim_{z \to \infty} X[z] = \lim_{z \to \infty} \left[x(0) + \frac{x(1)}{z} + \cdots \right]$$

$$= x[0] + \frac{x[1]}{\infty} + \cdots$$

$$= x[0]$$

$$\boxed{x[0] = \lim_{z \to \infty} X[z]}$$

13 **Final value Theorem:**

$$x[n] \rightleftharpoons X(z)$$

$$x[n]\big|_{n=\infty} = x(\infty) = \lim_{z \to 1} \{X(z)(1-z^{-1})\}$$

Conditions

1) $x[n] = 0, \; n < 0$

2) $(1-z^{-1}) X[z]$ should have poles inside unit circle in z-plane.

1. Find the z- transform $X(z)$ and sketch the pole-zero plot with ROC for each of the following sequences

a) $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[n]$

b) $x[n] = \left(\frac{1}{3}\right)^n u[n] + \left(\frac{1}{2}\right)^n u[-n-1]$

c) $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[-n-1]$

a) $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] . z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\left(\frac{1}{2}\right)^n . u[n] + \left(\frac{1}{3}\right)^n u[n] \right) . z^{-n}$$

for unit step sequence n exist at 0 to ∞

So . $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n . z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n u(n)$

by Expanding

$$= 1 + \frac{1}{2} z^{-1} + \left(\frac{1}{2} z^{-1}\right)^2 + \cdots$$

$$+ 1 + \frac{1}{3} z^{-1} + \left(\frac{1}{3} z^{-1}\right)^2 + \cdots$$

Common ratio's are $\frac{1}{2} z^{-1}$ & $\frac{1}{3} z^{-1}$

$$\text{So} = \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{1}{1 - \frac{1}{3} z^{-1}}$$

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$$= \frac{z}{z - 1/2} + \frac{z}{z - 1/3}$$

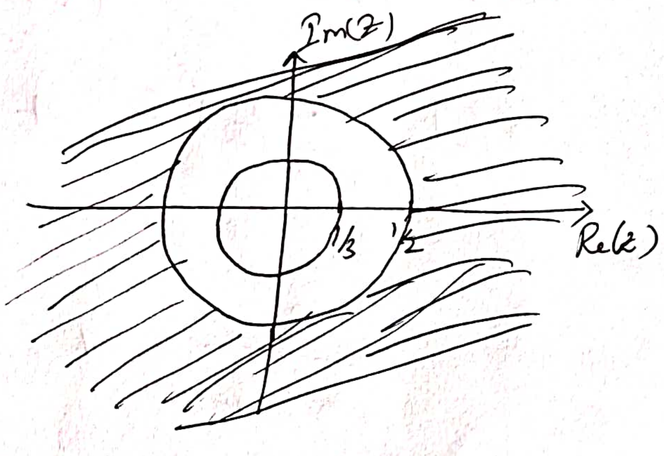
the $|z| > 1/2$ for $\frac{z}{z - 1/2}$

$|z| > 1/3$ for $\frac{z}{z - 1/3}$

we see the overlap.

$$X(z) = \frac{z}{z - 1/2} + \frac{z}{z - 1/3}$$

$$= \frac{2z(z - 5/12)}{(z - 1/2)(z - 1/3)} \quad |z| > 1/2$$



b) $x[n] = \left(\frac{1}{3}\right)^n u[n] + \left(\frac{1}{2}\right)^n u[-n-1]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

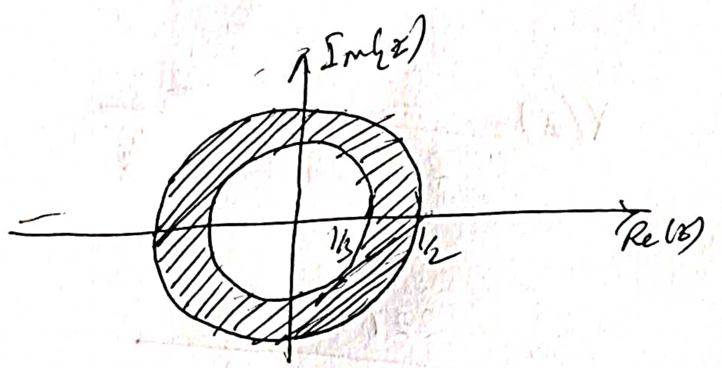
$$= \sum_{n=-\infty}^{\infty} \left(\left(\frac{1}{3}\right)^n u(n) + \left(\frac{1}{2}\right)^n u(-n-1) \right) . z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{3} z^{-1}\right)^n + \sum_{n=-\infty}^{-1} \left(\frac{1}{2} z^{-1}\right)^n$$

$$= \frac{z}{z - 1/3} + \frac{z}{z - 1/2}$$

$|\frac{1}{3} z^{-1}| < 1 \qquad |\frac{1}{2}^{-1} z| < 1$

$|z| > 1/3 \qquad |z| < 1/2$



(5) $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[n-1]$

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} + \sum_{n=-\infty}^{-1} \left(\frac{1}{3}\right)^n z^{-n}$$

$$= \frac{z}{z - 1/2} + \frac{z}{z - 1/3}$$

$|\frac{1}{2} z^{-1}| < 1 \qquad |\frac{1}{3}^{-1} z| < 1$

$|z| > 1/2 \qquad |z| < 1/3$

No common area found. According to ROC properties. No ROC will be found

2. Find the z-transform of the sequence

$$x[n] = n\left(-\frac{1}{2}\right)^n u[n] * \left(\frac{1}{4}\right)^{-n} u[-n]$$

Sol:

$$x[n] = n\left(\frac{-1}{2}\right)^n . u[n] * \left(\frac{1}{4}\right)^{-n} u[-n]$$

$\downarrow$

Solve this first

$$\left(\frac{-1}{2}\right)^n u[n] = \frac{z}{z+\frac{1}{2}} \quad \text{with ROC } |z| > \tfrac{1}{2}$$

$$w(n) = n.\left(\frac{-1}{2}\right)^n u(n) \longrightarrow -z\frac{d}{dz}\left(\frac{z}{z+\frac{1}{2}}\right)$$

$$= -z\left[\frac{1}{z+\frac{1}{2}} - \frac{z}{(z+\frac{1}{2})^2}\right]$$

$$= -z\left[\frac{\cancel{z}+\frac{1}{2}-\cancel{z}}{(z+\frac{1}{2})^2}\right]$$

$$= \frac{-\frac{1}{2}z}{(z+\frac{1}{2})^2}$$

Second term

$$y(n) = \left(\frac{1}{4}\right)^{-n} u[-n] \longrightarrow \frac{z^{-1}}{z^{-1}-\frac{1}{4}} = \frac{-4z}{z-4}$$

$$x[n] = w(n) * y(n) \xleftrightarrow{z} X(z) = W(z).Y(z)$$

$$= \frac{-\frac{1}{2}z}{(z+\frac{1}{2})^2} . \frac{-4z}{(z-4)} = \frac{2z^2}{(z-4)(z-\frac{1}{2})^2}$$

Where ROC is $\frac{1}{2} < |z| < 4$

3. $x[n] = a^n \cos(\omega_0 n)u[n]$ find the z. transform

Sol:

$$y[n] = a^n u[n)$$

$$Y(z) = \frac{1}{1-az^{-1}}$$

$$w[n] = \cos\omega_0 n = \frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}$$

$$x[n] = y(n).w(n)$$

$$= \frac{1}{2}\left(e^{j\omega_0 n}.y(n) + e^{-j\omega_0 n}.y(n)\right)$$

by applying complex exponential property to each term

$$X(z) = \frac{1}{2}Y(e^{-j\omega_0}z) + \frac{1}{2}Y(e^{j\omega_0}z) \quad \text{with ROC } |z| > a$$

$$= \frac{1}{2}\left(\frac{1}{1-ae^{j\omega_0}z^{-1}} + \frac{1}{1-ae^{-j\omega_0}z^{-1}}\right)$$

$$= \frac{1}{2}\left(\frac{1-ae^{-j\omega_0}z^{-1} + 1-ae^{j\omega_0}z^{-1}}{(1-ae^{j\omega_0}z^{-1})(1-ae^{-j\omega_0}z^{-1})}\right)$$

$$= \frac{1-a\cos\omega_0 z^{-1}}{1-2a\cos\omega_0 z^{-1}+a^2z^{-2}} \quad \text{with ROC } |z| > a$$

8

4. $x[n] = a^n \sin(\omega_0 n) u(n)$ find the z-transform

Sol: $x[n] = w(n) \cdot y(n)$

$$y(n) = a^n u(n) \rightarrow Y(z) = \frac{1}{1-az^{-1}}$$

$$\sin n\omega_0 = \frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2}$$

$$x[n] = \frac{1}{2}\left(a^n e^{j\omega_0 n} - e^{-j\omega_0 n} a^n\right) \cdot u(n)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} a^n e^{j\omega_0 n} z^{-n} - \sum_{n=0}^{\infty} a^n e^{-j\omega_0 n} z^{-n}$$

$$= \frac{1}{2}\left(\frac{1}{1-ae^{j\omega_0} z^{-1}} - \frac{1}{1-ae^{-j\omega_0} z^{-1}}\right)$$

$$= \frac{1}{2}\left(\frac{1-ae^{-j\omega_0} z^{-1} - 1 + ae^{+j\omega_0} z^{-1}}{(1-ae^{j\omega_0} z^{-1})(1-ae^{-j\omega_0} z^{-1})}\right)$$

$$= \frac{a \sin \omega_0 z^{-1}}{1 - 2a\sin\omega_0 z^{-1} + a^2 z^{-2}}$$ ROC $|z| > a$

5. Find the z- transform and the associated ROC for each of the following sequence

(a) $x[n] = \delta(n-n_0)$

(b) $x[n] = u[n-n_0]$

(c) $x[n] = a^{n+1} \cdot u[n+1]$

(d) $x[n] = u[-n]$

(e) $x[n] = a^{-n} \cdot u[-n]$

(a)

$$\delta[n] \longleftrightarrow 1$$

$$\delta[n-n_0] \longleftrightarrow z^{-n_0} \qquad 0 < |z|,\ n_0 > 0 \qquad |z| < \infty,\ n < 0$$

(b)

$$u[n] = \frac{z}{z-1}$$

$$u[n-n_0] \longleftrightarrow z^{-n_0} \cdot \frac{z}{z-1} = z^{-(n_0-1)}$$

(c)

$$a^n u[n] \longleftrightarrow \frac{z}{z-a} \qquad |z| > |a|$$

$$a^{n+1} u[n+1] \longleftrightarrow z \cdot \frac{z}{z-a} = \frac{z^2}{z-a}$$

(d)

$$u[n] \longleftrightarrow \frac{z}{z-1} \qquad |z| > 1$$

$$u[-n] = \frac{\frac{1}{z}}{\frac{1}{z}-1} = \frac{1}{1-z} \qquad |z| < 1$$

(e)

$$a^n u[n] \longleftrightarrow \frac{z}{z-a}$$

$$a^{-n} u[-n] \longleftrightarrow \frac{1/z}{1/z - a} = \frac{1}{1-az} \qquad |z| < \frac{1}{|a|}$$

6. Find the z- transform of each of the following sequen

(a) $x[n] = n.a^n.u[n]$

(b) $x[n] = n.a^{n-1}.u[n]$

Sol: a) $a^n.u[n] \longleftrightarrow \frac{z}{z-a} \quad |z|>|a|$

$$n.a^n.u[n] \longleftrightarrow -z\frac{d}{dz}\left(\frac{z}{z-a}\right)$$

$$= -z\left(\frac{1}{z-a} - \frac{z}{(z-a)^2}\right)$$

$$= -z\left[\frac{\cancel{z}-a-\cancel{z}}{(z-a)^2}\right]$$

$$= \frac{az}{(z-a)^2} \quad |z|>|a|$$

b) $$n\,a^{n-1}u[n] \longleftrightarrow \frac{d}{dz}\left(\frac{z}{z-a}\right) = \frac{z}{(z-a)^2} \quad |z|>|a|$$

$\underline{7}$ Find the inverse z-transform of

$$X(z) = \log\left(\frac{1}{1-az^{-1}}\right), \quad |z| < |a|$$

Sol:

$$X(z) = \sum_{n=1}^{\infty} \frac{1}{n}(a^{-1}z)^n$$

$$= \sum_{n=-1}^{-\infty} \frac{-1}{n} a^{n} z^{-n}$$

$$x[n] = \begin{cases} 0 & n \geq 0 \\ -(1/n)\, a^n & n \leq -1 \end{cases}$$

$$x[n] = -\frac{1}{n} a^n\, u[-n-1]$$

If $X(z) = \log\left(\frac{1}{1-az^{-1}}\right), \quad |z| > |a|$

The power ~~exp~~ series expansion for $\log(1-r)$

$$\log(1-r) = -\sum_{n=1}^{\infty} \frac{1}{n} r^n \qquad |r| < 1$$

Now

$$X(z) = \log\left(\frac{1}{1-az^{-1}}\right) = -\log(1-az^{-1}) \qquad |z| > |a|$$

Since $|z| > |a|$

$|az^{-1}| < 1$

So,

$$X(z) = \sum_{n=1}^{\infty} \frac{1}{n}(az^{-1})^n = \sum_{n=1}^{\infty} \frac{1}{n} a^n z^{-n}$$

$$x[n] = \begin{cases} (1/n)\, a^n & n \geq 1 \\ 0 & n \leq 0 \end{cases}$$

$$x[n] = \frac{1}{n} a^n u[n-1]$$

8 Using the power series expansion technique, find the inverse z-transform of the following.

$$X(z) = \frac{z}{2z^2 - 3z + 1} \qquad |z| < 1/2$$

$$X(z) = \frac{z}{2z^2 - 3z + 1} \qquad |z| > 1$$

a)

$$X(z) = \frac{z}{2z^2 - 3z + 1} \qquad |z| < 1/2$$

Since $|z| < 1/2$, $x[n]$ is a left sided sequence. Thus we must divide to obtain a series in power of z.

Carrying out the long division, we obtain

$$\begin{array}{r|l}
 & z + 3z^2 + 7z^3 + 15z^4 + \cdots \\
\hline
1 - 3z + 2z^2 & z \\
 & \cancel{z} - 3z^2 + 2z^3 \\
 & - \quad + \quad - \\
\hline
 & 3z^2 - 2z^3 \\
 & \cancel{3z^2} - 9z^3 + 6z^4 \\
 & - \quad + \quad - \\
\hline
 & 7z^3 - 6z^4 \\
 & \cancel{7z^3} - 21z^4 + 14z^5 \\
 & - \quad + \quad - \\
\hline
 & 15z^4 - 14z^5 \\
 & \cancel{15z^4} - 45z^5 + 30z^6 \\
 & - \quad + \quad - \\
\hline
 & 31z^5 - 30z^6
\end{array}$$

$X(z) = \dots + 15z^4 + 7z^3 + 3z^2 + z$

So.

$x[n] = \{\dots, 15, 7, 3, 1, \underset{\uparrow}{0}\}$

b) Since ROC is $|z| > 1$. $x[n]$ is right sided sequence.

We must divide so as to obtain power series as z^{-1}

$$
\begin{array}{r|l}
 & \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} + \dots \\
\hline
2z^2 - 3z + 1 & \cancel{z} \\
 & \cancel{z} - \frac{3}{2} - \frac{1}{2}z^{-1} \\
 & \quad - \quad + \quad - \\
\hline
 & \frac{3}{2} - \frac{1}{2}z^{-1} \\
 & \cancel{\frac{3}{2}} - \frac{9}{4}z^{-1} + \frac{3}{4}z^{-2} \\
 & \quad - \quad + \quad - \\
\hline
 & \frac{7}{4}z^{-1} - \frac{3}{4}z^{-2} \\
 & \vdots
\end{array}
$$

$X(z) = \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} + \frac{15}{16}z^{-4} + \dots$

So,

$x[n] = \{\underset{\uparrow}{0}, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \dots\}$

Q Find the inverse z-transform of

$X(z) = \dfrac{z}{z(z-1)(z-2)^2} \qquad |z| > 2$

Sol:

$\dfrac{X(z)}{z} = \dfrac{1}{z(z-1)(z-2)^2}$

$$\frac{1}{z(z-1)(z-2)^2} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{(z-2)} + \frac{D}{(z-2)^2}$$

$$A(z-1)(z-2)^2 + B(z-2)^2 z + C.z(z-2)(z-1)$$
$$+ D.z.(z-1) = 1 \quad \text{—} (1)$$

put $z=0$

$$A(-1)(-2)^2 = 1$$

$$A = -1/4$$

Compare like terms

$$A+B+C=0$$
$$-3A+2B+3C+D=0$$
$$A+B+2C-D=0$$

$$-2A+3B-C=0$$
$$A+B+C=0$$

$$-A+4B=0$$

$$A = 4B$$
$$B = \frac{A}{4} = -1/4 \times \frac{1}{4}$$
$$= -1/16$$

$$A+B+C=0$$
$$\frac{-1}{4} - \frac{1}{16} + C = 0$$
$$C = \frac{5}{16}$$

$$A+B+2C-D=0$$
$$D = A+B+2C$$
$$= \frac{-1}{4} - \frac{1}{16} + \frac{2\times 5}{16}$$
$$= \frac{-4-1+10}{16} = 5/16$$

$$\frac{X(z)}{\cancel{z}} = \frac{-1}{4}\times\frac{1}{z} + \frac{-1}{16}\cdot\frac{1}{z-1} + \frac{5}{16}\cdot\frac{1}{(z-2)} + \frac{5}{16}\cdot\frac{1}{(z-2)^2}$$

$$X(z) = \frac{-1}{4}\times\frac{\cancel{z}}{\cancel{z}} - \frac{1}{16}\cdot\frac{z}{z-1} + \frac{5}{16}\cdot\frac{z}{z-2} + \frac{5}{16}\cdot\frac{z}{(z-2)^2}$$

$$X(z) = \frac{-1}{4} - \frac{1}{16}\cdot\frac{z}{z-1} + \frac{5}{16}\cdot\frac{z}{(z-2)} + \frac{5}{16}\cdot\frac{z}{(z-2)^2}$$

Inverse z-transform

$$= \frac{-1}{4}\delta[n] - \frac{1}{16}\cdot(1)^n\cdot u(n) + \frac{5}{16}\cdot 2^n\cdot u(n) + \frac{5}{16}n\cdot 2^{n-1}\cdot u(n)$$

$$= \frac{-1}{4}\delta(n) - \left(\frac{1}{16} + \frac{5}{16}2^n + \frac{5}{16}n\cdot 2^{n-1}\right)u(n)$$

10) find the inverse z-transform of $X(z) = \frac{2z^3-5z^2+z+3}{(z-1)(z-2)}$

where ROC is $|z| < 1$

Sol:

$$X(z) = \frac{2z^3-5z^2+z+3}{(z-1)(z-2)} = \frac{2z^3-5z^2+z+3}{z^2+2-3z}$$

$X(z)$ is improper rational function

So.

$$
\begin{array}{r|l}
 & \quad 2z+1 \\
z^2-3z+2 & \cancel{2z^3}-5z^2+z+3 \\
 & \cancel{2z^3}-6z^2+4z \\
 & \;-\quad +\quad - \\
\hline
 & \quad \cancel{z^2}-\cancel{3z}+3 \\
 & \quad \cancel{z^2}-\cancel{3z}+2 \\
 & \quad -\quad +\quad - \\
\hline
 & \qquad\quad 1
\end{array}
$$

So. we can write as

$$X(z) = 2z+1+\underbrace{\frac{1}{(z-1)(z-2)}}_{W(z)}$$

Now

$$\frac{W(z)}{z} = \frac{1}{z(z-1)(z-2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2}$$

$$A(z-1)(z-2) + B(z\cdot(z-2)) + C(z\cdot(z-1)) = 1$$

~~A~~ Comparing like terms (or) by substitute $z = 0, 1, 2$ the A, B, C values can be evaluated putting $z = 0$

$$A(0-1)(0-2) + B \cdot 0 \cdot + C \cdot 0 = 1$$

$$A = 1/2$$

putting $z = 1$

$$A(1-1)(1-2) + B(1)(1-2) + C(1)(1-1) = 1$$

$$0 - B + 0 = 1$$

$$B = -1$$

putting $z = 2$

$$A \cdot 0 \cdot 1 + B(1)(0) + C(2)(1) = 1$$

$$\underline{\underline{C = 1/2}}$$

$$\frac{W(z)}{z} = \frac{1}{2 \cdot z} - \frac{1}{z-1} + \frac{1}{2(z-2)}$$

$$W(z) = \frac{1}{2} - \frac{z}{z-1} + \frac{z}{2(z-2)}$$

$$X(z) = 2z + 1 + \frac{1}{2} - \frac{z}{z-1} + \frac{z}{2(z-2)}$$

$$= 2z + \frac{3}{2} - \frac{z}{z-1} + \frac{1}{2}\frac{z}{(z-2)}$$

by IZT

$$= 2\delta(n+1) + \frac{3}{2}\delta(n) + 1 \cdot u[-n-1] - \frac{1}{2}2^n u[n-1]$$

$$= \underline{\underline{\left(2\delta(n+1) + \frac{3}{2}\delta(n) + (1 - 2^{n-1})u[-n-1]\right)}}$$

11. Using Residue method, find the inverse z-transform of

$$X(z) = \frac{1+2z^{-1}}{1+4z^{-1}+3z^{-2}} \; ; \; ROC; \; |z| > 3$$

Sol:

$$X(z) = \frac{1+2z^{-1}}{1+4z^{-1}+3z^{-2}} = \frac{z(z+2)}{z^2+4z+3} = \frac{z(z+2)}{(z+1)(z+3)}$$

$x(n) = \sum$ Residues of $X(z)z^{n-1}$ at the poles of $X(z)z^{n-1}$ within C

$= \sum$ Residues of $\frac{z(z+2)z^{n-1}}{(z+1)(z+3)} = \frac{z^n(z+2)}{(z+1)(z+3)}$ at poles of the

$= \sum$ Residues of $\frac{z^n(z+2)}{(z+1)(z+3)}$ at poles $z=-1$ and $z=-3$

$$= \left.(z+1)\frac{z^n(z+2)}{(z+1)(z+3)}\right|_{z=-1} + \left.\frac{(z+3)(z+2)z^n}{(z+1)(z+3)}\right|_{z=-3}$$

$$= \frac{z^n(-1+2)}{-1+3} + \frac{(-3)^n(-3+2)}{-3+1}$$

$$= \frac{1}{2}(-1)^n u(n) + \frac{1}{2}(-3)^n u(n)$$

inverse z transm of $X(z)$

$$x(n) = \frac{1}{2}(-1)^n u(n) + \frac{1}{2}(-3)^n u(n)$$

12 Determin the inverse z-transform using complex integral

$$X(z) = \frac{3z^{-1}}{(1-(1/2)z^{-1})^2} ; \quad ROC ; |z| > 1/2$$

Sol:

$$x(n) = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

This can be evaluated by finding the sum of all residues of the poles that are inside the circle.

$$x(n) = \sum \text{residues of } X(z) z^{n-1} \text{ at the poles}$$

$$= \sum_i (z-z_i) X(z) z^{n-1} \Big|_{z=z_i}$$

$$= \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left[(z-z_i)^k X(z) z^{n-1} \right] \text{ at the pole } z=z_i$$

$$X(z) = \frac{3z^{-1}}{[1-(1/2)z^{-1}]^2} = \frac{3z}{(z-1/2)^2}$$

order = 2 , z = 1/2

$$x(n) = \sum \text{residues of } X(z) z^{n-1} \text{ at its poles}$$

$$x(n) = \frac{1}{1!} \frac{d}{dz} \left(\left(z-\frac{1}{2}\right)^2 \frac{3z^n}{[z-(1/2)]^2} \right) \Bigg|_{z=1/2} = 3nz^{n-1} \Big|_{z=1/2}$$

$$= 3n\left(\frac{1}{n}\right)^{n-1} u(n)$$